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Distance-Based Similarity Measures in T-Spherical Fuzzy Hypersoft Environment for Cluster Analysis

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ABSTRACT

The notion of T-spherical fuzzy hypersoft set (TSFHSS) provides a powerful generalization of the picture fuzzy hypersoft set, offering additional flexibility in modeling uncertain and unreliable information where the sum of membership, non-membership, and neutrality degrees can exceed one. The concept of similarity/distance measures plays a crucial role in various pattern recognition, decision-making, classification, and cluster analysis tasks. In the present study, we introduce a novel distance measure for T-spherical fuzzy hypersoft sets to effectively quantify the dissimilar aspects of two objects. The proposed measures satisfy the necessary axioms and possess several desirable mathematical properties. Based on these measures, we define corresponding similarity indices that are particularly effective in distinguishing highly similar elements and improving discrimination among objects. To validate their practical usefulness, a numerical illustration is presented demonstrating their accuracy and stability. Furthermore, we develop a clustering algorithm that utilizes the introduced similarity measures under the TSFHSS framework. The results indicate that the proposed TSFHSS-based measures enhance clustering performance by providing better discrimination capability and more reliable grouping of objects in uncertain environments compared with conventional approaches.

1. Introduction

Various real-world issues, like clustering, sequence alignment, and medical diagnosis, generates the necessity to a certain level of analogy among the items. Diversification of compatibility metrics may be suitable for various problems related to the field of pattern recognition. The predominant comparison metrics include similarity/distance measures which has broader applications in various

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engineering problems. Almost all the facet of our daily existence encompasses a degree of uncertainty. The notion of similarity has been explored in various extended set-theory structures which has the capability to address inconsistency/inexactness in the information, given that the aforementioned difficulties pertain to multiple forms of uncertainty. To address these uncertainties, Zadeh [31] defined fuzzy set (FS) in 1965 filling the gap between real world and actual mathematical compositional structures. The set is characterized by a membership function where the value of belongingness ranging from 0 to 1. This proposed structure was then extended by different researchers, by proposing Intuitionistic Fuzzy Sets (IFS) [3], Picture Fuzzy Sets (PFSs) [6], and T-spherical Fuzzy Sets (TSFSs) [2]. Additionally, numerous similarity metrics on picture fuzzy sets have been utilized in various decision-making scenarios [1], [20], [21], [25]. Subsequently, Molodtsov [17] presented soft set theory for addressing MCDM problems in case where uncertain issues are in a parametric context. Garg and Arora devised various types of aggregation operators (AOs) where the correlation coefficient and TOPSIS approach for IFSs [12], [13] have been taken into consideration. In 2018, Smarandache [23] proposed hypersoft set as a further extension of soft set (SS) theory that transforms the function into a multi-structured function, proving highly beneficial for MCDM situations. The concept of fuzzy hypersoft set (FHS) [23], integrated the idea of fuzzy set in perspective of hypersoft set. Adem Yolcu et al. [30] introduced an Intuitionistic Fuzzy Hypersoft Set (IFHSS) and explained many operations, e.g., subset, union, intersection, complement, AND, and OR operations, to address the challenge of uncertainty with greater precision. Dhumras and Bajaj [7], [11] introduced an enhanced PFHSSs in MCDM context incorporating the degrees of abstention and refusal, along with a sub-parametrizations in case of attributes. Furthermore, similarity metrics for PFHSSs have been effectively utilized in medical diagnostic issues [9]. The notion of TSFHSS was introduced by P. Bansal et al. [5], offering the additional benefit of permitting elements without limits, hence eliminating the constraint of a sum of one on the uncertainty components. Monika et al. [18] have examined various aggregation operators (arithmetic and geometric) for TSFHSS sets, highlighting their multiple features and yielding significant results for renewable energy sources. The similarity metric is crucial for assessing the levels of relationship between two or more elements. The preliminary research on similarity metrics for FS was conducted by Wang [27] in 1982. Subsequently, other researchers have conducted studies on similarity measures in feature selection. Following numerous investigations into the entropy/distance/similarity measures of fuzzy sets [29], [15], these metrics were subsequently expanded to include intuitionistic fuzzy sets [16], picture fuzzy sets [19], and others. Athira et al. [4] expanded similarity measures for Pythagorean fuzzy soft sets (PyFSSs) and employed their newly formulated similarity metrics for cluster analysis. Garg and Nancy [14] investigated a decision science approach utilizing a distance measure to address SVNS-MCDM problems through the application of neutrosophic set. Novel similarity metrics for PFSs, as defined by S. Singh and Ganie [22], with applications in pattern recognition, clustering, pattern recognition and MADM problems. Recent similarity measures have been introduced for neutrosophic hypersoft sets and PFHSs, accompanied by diverse MCDM application probelms [8], [10]. The correlation coefficient for T-spherical fuzzy sets proposed by Ullah et al. [26] includes its properties and establishes new methodologies for addressing clustering issues and multi-attribute decision-making challenges inside the TSFS framework. Thao [24] introduced a novel correlation coefficient for PFSs, ranging from [1, 1], to address the inverse correlation between PFSs and extended this methodology to compute the correlation coefficient between IVPFSs.

In recent years, several extensions of classical fuzzy sets have been developed to better handle uncertainty in complex decision environments. While fuzzy sets capture vagueness through membership grades, intuitionistic fuzzy sets incorporate both membership and non-membership information, thereby improving descriptive power. Subsequent models such as picture fuzzy sets and hypersoft sets further enhance flexibility by allowing additional opinion dimensions and multi-parameter attribute representation. However, these frameworks still impose certain structural restrictions on the admissible information space. The TSFHSS model overcomes these limitations by combining the parameterized structure of hypersoft sets with the enlarged feasible domain of T-spherical fuzzy information, enabling more expressive representation of uncertainty. Owing to this enhanced modeling capability, TSFHSS provides a more suitable foundation for developing similarity measures and clustering tools in situations involving complex, multi-attribute, and imprecise data. The overall framework of the proposed study, including the development of similarity measures under T-spherical fuzzy hypersoft set structures and their applications in decision-making problems, is illustrated in Figure 1.

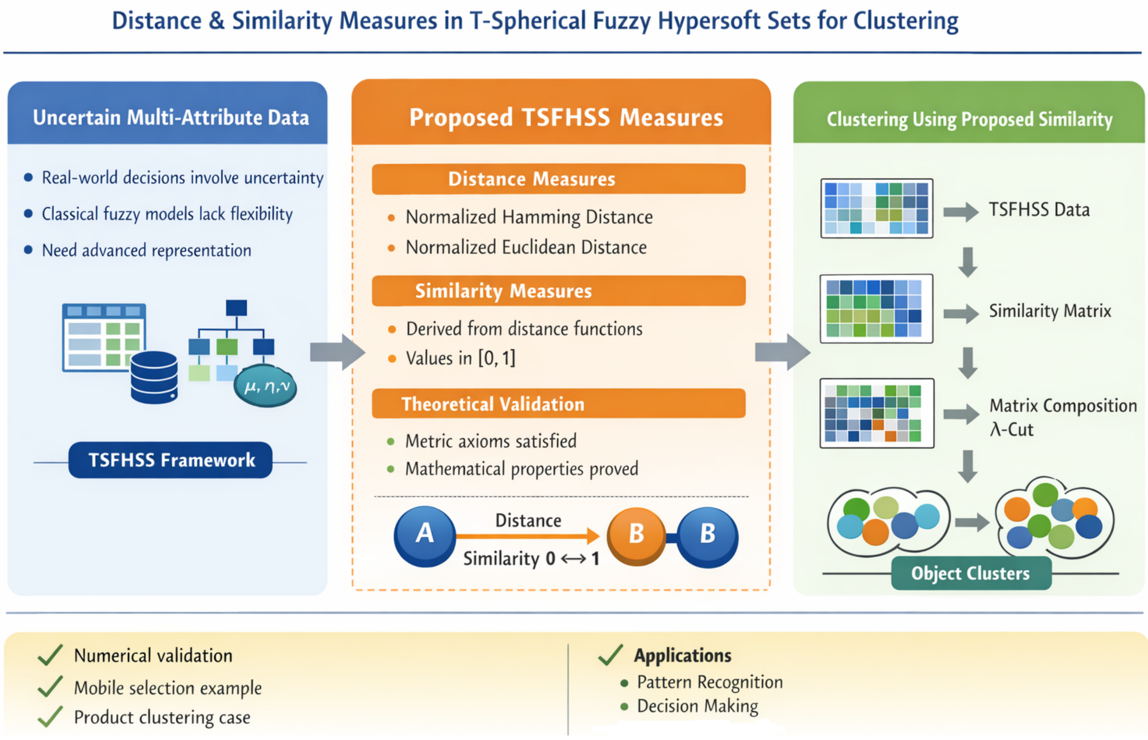

Figure 1: Graphical representation of the proposed framework

The remainder of the manuscript is structured as follows: Section 2 provides fundamental definitions pertinent to the present study. Distance and similarity measurements for $TSFHSS$ are proposed in Section 3. Certain theorems regarding measurements are formulated to articulate their qualities. A numerical example is also supplied to verify the accuracy of the proposed measures. A clustering technique founded on the similarity metrics delineated in Section 4 and exemplified through a numerical case study. Finally, Section 5 provides a comprehensive conclusion along with the potential for further research.

2. Preliminaries

Some fundamental definitions which are useful for the discussions in the present communication are included in this section. The standard available notions in literature for T-spherical fuzzy sets, hypersoft sets, and T-spherical fuzzy hypersoft sets, distance/similarity measures are presented for better readability and understanding.

Definition 1. *"T-**Spherical Fuzzy Set (TSFS) [2]**. A T-Spherical Fuzzy Set A in U is $A = \{u, \rho_A(u), \tau_A(u), \eta_A(u) : u \in U\}$ where, $\rho_A : U \to [0,1]$ is the degree of positive membership of u in A, $\tau_A : U \to [0,1]$ is the degree of neutral membership of u in A, $\eta_A : U \to [0,1]$ is the degree of negative mem-*

bership of u in A and, $\rho_A,\ \tau_A,\ \eta_A$ satisfies the constraint ;

$$\rho_A^n(u) + \tau_A^n(u) + \eta_A^n(u) \le 1\ (\forall\, u \in U);$$

and $\pi_A(u) = \sqrt{1 - (\rho_A^n(u) + \tau_A^n(u) + \eta_A^n(u))}$ is called the degree of refusal membership of u in U. We denote the set of all the T-spherical fuzzy sets over U by $TSFS(U)$."

Definition 2. *"**Hypersoft Set (HSS) [23].** Let U be the universal set and P(U) be the power set of U. Consider $e_1, e_2, \ldots . e_n$ for $n \ge 1$, be n well-defined attributes, whose corresponding attribute values are the sets $E_1, E_2, \ldots, E_n$ with $E_i \cap E_j = \varphi$ for $i \ne j$ and $i, j \in \{1, 2, \ldots, n\}$.*

Then the pair $(R, E_1 \times E_2 \times \ldots . E_n)$ is said to be Hypersoft Set over U where $R : E_1 \times E_2 \times \ldots . \times E_n \to P(U)$. In other words, Hypersoft Set is a multi-parameterized family of subsets of the set U."

Definition 3. *"**T-Spherical Fuzzy Hypersoft Set (TSFHSS) [5].** Let U be the universal set and TSFHSS(U) be the set of all T-Spherical fuzzy subsets of U. Consider $e_1, e_2, \ldots . e_n$ for $n \ge 1$, be n well-defined attributes, whose corresponding attribute values are the sets $E_1, E_2, \ldots, E_n$ with $E_i \cap E_j = \varphi$ for $i \ne j$ and $i, j \in \{1, 2, \ldots, n\}$. Let A_i be the non-empty subsets of E_i for each $i = 1, 2, \ldots, n$.*
A TSFHSS is defined as the pair
$(R, A_1 \times A_2 \times \ldots . \times A_n)$; where $R : E_1 \times E_2 \times \ldots \times E_n \to \mathrm{T-SFS}\,(U)$ and

$$R\,(\xi) = \left\{ < \xi, \left(\frac{u}{\rho_{R(\xi)}\,(u)\,, \tau_{R(\xi)}\,(u)\,, \eta_{R(\xi)}(u)} \right) >:\ u\ \in\ U \right\}.$$

It may be noted that $\xi \in\ A_1 \times A_2 \times \ldots . \times A_n \subseteq E_1 \times E_2 \times \ldots . E_n$ and $\rho\,, \tau$ and η represent the positive membership, neutral membership and negative membership degrees respectively."

Definition 4. *Let $\Psi(U)$ be collection of all TSFHSS over universal set U. A mapping $\phi : \Psi(U) \times \Psi(U) \to [0, 1]$ is termed as distance similarity measures, if it satisfies the following axioms for $R_1, R_2, R_3 \in \Psi(U)$.*

(D1) : $0 \le \phi(R_1, R_2) \le 1$;

(D2) : $\phi(R_1, R_2) = 0 \Leftrightarrow R_1 = R_2$;

(D3) : $\phi(R_1, R_2) = \phi(R_2, R_1)$;

(D4) : *If $R_1 \subseteq R_2 \subseteq R_3$ then $\phi(R_1, R_2) \le \phi(R_1, R_3)$ and $\phi(R_2, R_3) \le \phi(R_1, R_3)$.*

3. Distance/Similarity Measures for T-SFHSS

We present a novel distance/similarity measure for TSFHSS in this section and study their important characteristics.

Definition 5. ***Normalization of Hamming Distance for TSFHSS.** Let R_1 and R_2 be two TSFHSS where $R_1 = (\rho_{R_1}(u_i), \tau_{R_1}(u_i), \eta_{R_1}(u_i))$ and $R_2 = (\rho_{R_2}(u_i), \tau_{R_2}(u_i), \eta_{R_2}(u_i))$.*
A normalized Hamming distance between R_1 and R_2 is defined as

$$\phi_1(R_1, R_2) = \frac{1}{3n} \sum_{i=1}^{n} \left(\left|\rho_{R_1}^n(u_i) - \rho_{R_2}^n(u_i)\right| + \left|\tau_{R_1}^n(u_i) - \tau_{R_2}^n(u_i)\right| + \left|\eta_{R_1}^n(u_i) - \eta_{R_2}^n(u_i)\right| \right).$$

Definition 6. ***Normalized Euclidean Distance for TSFHSS.** Let R_1 and R_2 be two TSFHSS where $R_1 = (\rho_{R_1}(u_i), \tau_{R_1}(u_i), \eta_{R_1}(u_i))$ and $R_2 = (\rho_{R_2}(u_i), \tau_{R_2}(u_i), \eta_{R_2}(u_i))$.*
A normalized Euclidean distance between R_1 and R_2 is defined as

$$\phi_2(R_1, R_2) = \sqrt{ \frac{1}{3n} \sum_{i=1}^{n} \left(\left|\rho_{R_1}^n(u_i) - \rho_{R_2}^n(u_i)\right|^2 + \left|\tau_{R_1}^n(u_i) - \tau_{R_2}^n(u_i)\right|^2 + \left|\eta_{R_1}^n(u_i) - \eta_{R_2}^n(u_i)\right|^2 \right). }$$

Theorem 1. *The above defined both measures ϕ_1 and ϕ_2 satisfies all the basic necessary axioms as per the definition 4.*

Proof. Consider two TSFHSS R_1 and R_2 defined over U.

(D1) For each i, $\rho_{R_1}(u_i), \tau_{R_1}(u_i), \eta_{R_1}(u_i) \in [0,1]$.
Similarly $\rho_{R_2}(u_i), \tau_{R_2}(u_i), \eta_{R_2}(u_i) \in [0,1]$ for n. Therefore, we have

$$\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right|, \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right|, \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right| \in [0,1].$$

Further we get,

$$0 \leq \frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|\right) \leq 1$$

which implies that

$$\phi_1(R_1, R_2) \in [0,1].$$

(D2) If $\phi_1(R_1, R_2) = 0$, which implies

$$\frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|\right) = 0.$$

$$\Leftrightarrow \sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|\right) = 0.$$

$$\Leftrightarrow \left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right|, \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right|, \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right| = 0.$$

From this, we get $\rho^n_{R_1}(u_i) = \rho^n_{R_2}(u_i)$, $\tau^n_{R_1}(u_i) = \tau^n_{R_2}(u_i)$, $\eta^n_{R_1}(u_i) = \eta^n_{R_2}(u_i)$ for each $i = 1, 2, ..., n$.
Hence we have,

$$R_1 = R_2.$$

(D3) For each $i = 1, 2, ..., n$ $\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right|, \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right|, \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|$ are symmetric therefore $\phi_1(R_1, R_2) = \phi_1(R_2, R_1)$.

(D4) To prove axiom $(D4)$ consider $R_1 \subseteq R_2 \subseteq R_3$ it implies that $\rho^n_{R_1}(u_i) \leq \rho^n_{R_2}(u_i) \leq \rho^n_{R_3}(u_i)$, $\tau^n_{R_1}(u_i) \geq \tau^n_{R_2}(u_i) \geq \tau^n_{R_3}(u_i)$ and $\eta^n_{R_1}(u_i) \geq \eta^n_{R_2}(u_i) \geq \eta^n_{R_3}(u_i) \forall, i$ then we get, $\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| \leq \left|\rho^n_{R_1}(u_i) - \rho^n_{R_3}(u_i)\right|$, $\left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| \leq \left|\tau^n_{R_1}(u_i) - \tau^n_{R_3}(u_i)\right|$, and $\left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right| \leq \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|$. Therefore,

$$\frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|\right)$$

$$\leq \frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_3}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_3}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_3}(u_i)\right|\right).$$

Hence,

$$\phi_1(R_1, R_2) \leq \phi_1(R_1, R_3).$$

On similar lines, the measure ϕ_2 fulfills the basic necessary conditions which implies the proof for the derivation. Further, the process of computing the similarity score on the basis of defined distance measure has been explained as follows. □

Definition 7. *The value of similarity score analogous to the normalization of Hamming distance is given below:*

$$S_h(R_1, R_2) = 1 - \frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right| + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right| + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|\right). \quad (1)$$

We can write as $S_h(R_1, R_2) = 1 - \phi_1(R_1, R_2)$.

Definition 8. *The value of similarity score analogous to the normalization of Euclidean distance is given below:*

$$S_e(R_1, R_2) = 1 - \sqrt{\frac{1}{3n}\sum_{i=1}^{n}\left(\left|\rho^n_{R_1}(u_i) - \rho^n_{R_2}(u_i)\right|^2 + \left|\tau^n_{R_1}(u_i) - \tau^n_{R_2}(u_i)\right|^2 + \left|\eta^n_{R_1}(u_i) - \eta^n_{R_2}(u_i)\right|^2\right)}. \quad (2)$$

We can write as $S_e(R_1, R_2) = 1 - \phi_2(R_1, R_2)$.

Theorem 2. *The measures defined in equation 1 and 2 satisfies*

(P1) $0 \leq S(R_1, R_2) \leq 1$;

(P2) $S(R_1, R_2) = 1$ *if* $R_1 = R_2$;

(P3) $S(R_1, R_2) = S(R_2, R_1)$;

(P4) *If* $R_1 \subseteq R_2 \subseteq R_3$ *then* $S(R_1, R_3) \leq S(R_1, R_2)$ *and* $S(R_1, R_3) \leq S(R_2, R_3)$.

Proof. Consider two TSFHSS R_1 and R_2, we have

(P1) Here, $0 \leq \phi_1(R_1, R_2) \leq 1$, which implies that $0 \leq 1 - \phi_1(R_1, R_2) \leq 1$ i.e. $0 \leq S(R_1, R_2) \leq 1$.

(P2) $S(R_1, R_2) = 1 \Leftrightarrow \phi_1(R_1, R_2) = 0$ if $R_1 = R_2$.

(P3) $S(R_1, R_2) = S(R_2, R_1)$ it results from the definition .

(P4) If $R_1 \subseteq R_2 \subseteq R_3$, by distance measures $\phi_1(R_1, R_3) \leq \phi_1(R_1, R_2)$ and $\phi_1(R_2, R_3) \leq \phi_1(R_1, R_3)$ which implies this $1 - \phi_1(R_1, R_3) \leq 1 - \phi_1(R_1, R_2)$ and $1 - \phi_1(R_1, R_3) \leq \phi_1(R_2, R_3)$. Hence $S(R_1, R_3) \leq S(R_1, R_2)$ and $S(R_1, R_3) \leq S(R_2, R_3)$.

The deliberations related to the measures under considerations have been shown with the help of a numerical example for value of $n = 3$: □

3.1 Numerical Illustration

Suppose in the market, a buyer wishes to purchase a mobile phone from the stalls. There are 4 different types of companies available and the buyers have their own selection criteria/objectives for the mobile phones available. Let $U = \{u_1, u_2, u_3, u_4\}$ be the set of four different type of mobiles brands with necessary information on the attributes as e_1 = Battery capacity, e_2 = RAM, e_3 = Price, e_4 = Colors and respective sub-attributes are $E_1 = \{4000mAh, 5000mAh, 6000mAh\}$, E_2 = {4GB, 6GB, 8GB}, $E_3 = \{20000, 30000, 40000\}$, $E_4 = \{White, Black, Grey\}$. Next, suppose that there are two TSFHSS say R_1 and R_2 where
$R_1\{5000mAh, 4GB, 20000, Black\} = \{u_1(0.8, 0.6, 0.2), u_2(0.4, 0.5, 0.7)\}$ and

$R_2\{5000mA, 4GB, 20000, Black,\} = \{u_1(0.6, 0.4, 0.5), u_2(0.9, 0.2, 0.2)\}$ then;
Normalized Hamming distance

$$= \frac{1}{3n}\left(\left|0.8^3 - 0.6^3\right| + \left|0.6^3 - 0.4^3\right| + \left|0.2^3 - 0.5^3\right| + \left|0.4^3 - 0.9^3\right| + \left|0.5^3 - 0.2^3\right| + \left|0.7^3 - 0.2^3\right|\right)$$

= 0.1402.

Normalized Euclidean distance

$$= \sqrt{\frac{1}{3n}\left(\left|0.8^3 - 0.6^3\right|^2 + \left|0.6^3 - 0.4^3\right|^2 + \left|0.2^3 - 0.5^3\right|^2 + \left|0.4^3 - 0.9^3\right|^2 + \left|0.5^3 - 0.2^3\right|^2 + \left|0.7^3 - 0.2^3\right|^2\right)}$$

= 0.058.
On similar lines, the similarity scores are computed as follows:

$$1 - \phi_1(R_1, R_2) = 1 - 0.1402 = 0.8598 \in [0, 1].$$

$$1 - \phi_2(R_1, R_2) = 1 - 0.058 = 0.942 \in [0, 1].$$

4. Distance/Similarity Measure in Clustering

Different studies and practical applications of the clustering analysis have been conducted across diverse domains in literature. In this section, we introduce the TSFHSS clustering algorithm, which is designed to group heterogeneous objects into homogeneous clusters based on the proposed similarity measures. Initially, we present several necessary definitions that are fundamental to the overall process. Subsequently, the detailed procedure of the proposed methodology is demonstrated in figure 2.

Definition 9. *Let (R_m) be m number of TSFHSS over U, a matrix $N = [N_{jk}]_{m\times m}$, where $N_{jk} = S(R_j, R_k), (j, k = 1, 2, ..., m)$ is called similarity matrix between the TSFHSS which satisfies the following properties:*

1. $0 \le N_{jk} \le 1$;
2. $N_{jj} = 1$;
3. $N_{jk} = N_{kj}$; *where* $j, k = 1, 2, ..., m$.

Definition 10. *[28] Let $N = [N_{jk}]_{m\times m}$ be a similarity matrix, if $N^2 = N \circ N = [N_{jk}]_{m\times m}$, then N^2 is composition matrix of N, where $N_{jk} = \max_p\{\min\{N_{jp}, N_{pk}\}\}$,
$j, k = 1, 2, ..., m$.*

Definition 11. *[28] For $N = [N_{jk}]_{m\times m}$ and for $N \to N^2 \to N^4 \to, ..., \to N^{2n} \to ...,$ there should have an integer $n > 0$ such that $N^{2n} = N^{2^{(n+1)}}$ and N^{2^n} is an equivalent similarity matrix.*

Definition 12. *[28] For $N = [N_{jk}]_{m\times m}$ $\lambda-$ cutting matrix of N is $N^\lambda = [N^\lambda_{jk}]_{m\times m}$, where*

$$N^\lambda_{jk} = \begin{cases} 1 & ; \quad N^\lambda_{jk} \ge \lambda \\ 0 & ; \quad N^\lambda_{jk} \le \lambda \end{cases} \tag{3}$$

and $\lambda \in [0, 1]$ is the confidential interval.

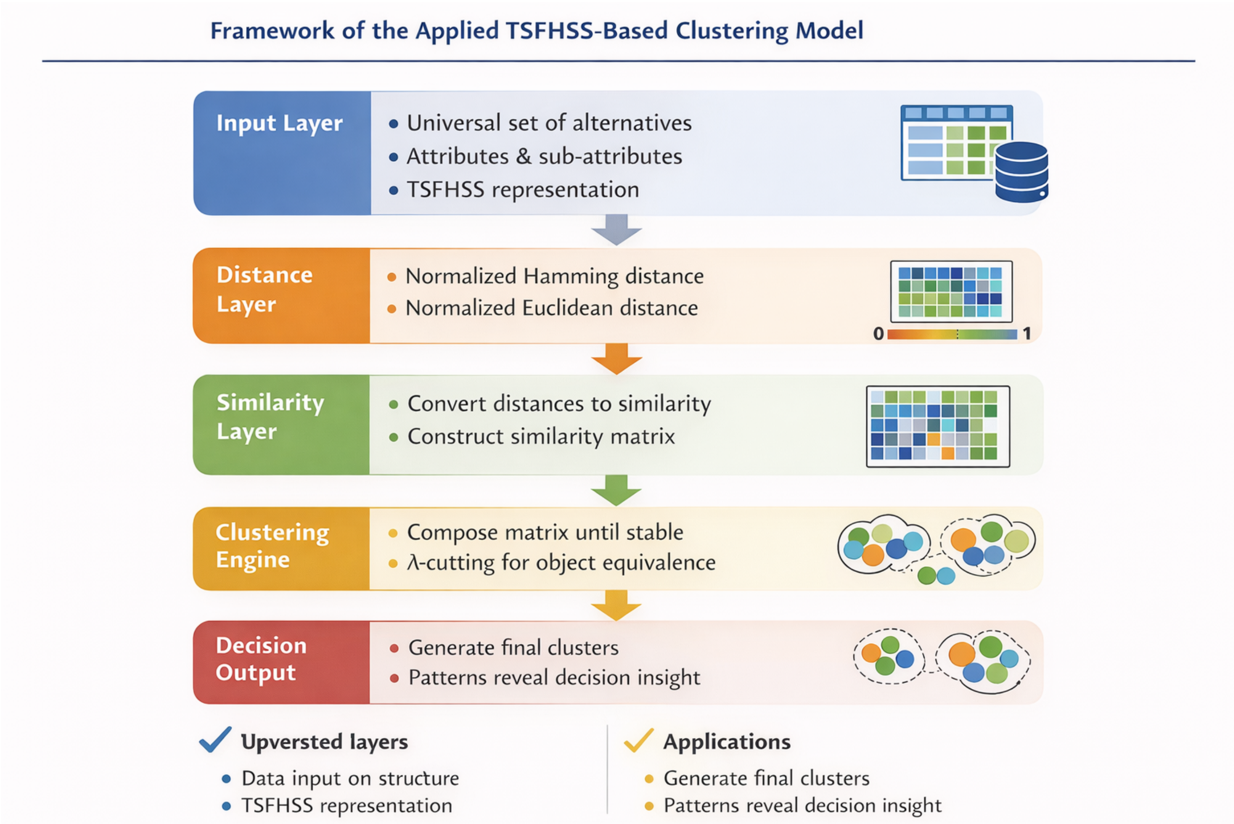

Figure 2: Step-by-step procedure of the proposed TSFHSSS-similarity based approach

Subsequently, a clustering technique on the basis of the measure has been presented. The decision-makers opinions on each individual alternatives $\{X_1, X_2, ..., X_m\}$ are provided with respect to n attribute in terms of TSFHSS. The purpose of this work is to classify the available X_j' into the classes as per the technique. We follow the suitable suggested steps to accomplish this.

Step (i): Construction of matrix $N = [N_{jk}]_{m\times m}$, where $N_{jk} = S(R_j, R_k), (j, k = 1, 2, ..., m)$ with the help of Equation (1).

Step (ii): Calculate N^{2^z}, where $z = 1, 2, , 3...$, where $N \rightarrow N^2 \rightarrow N^4 \rightarrow, ..., \rightarrow N^{2^z} \rightarrow ...$, until $N^{2^z} = N^{2^{(z+1)}}$.

Step (iii): Formulate the $\lambda-$cutting matrix $N^\lambda = [N^\lambda_{jk}]_{m\times m}$, with the help of definition 12.

Step (iv): X_j and X_k are classified in the same group if they are identical.

Numerical Illustration

Consider five different class of refrigerators, e.g., $\{X_1, X_2, X_3, X_4, X_5\}$ their corresponding attributes are $\{e_1, e_2, e_3, e_4\}$ denoting the price, type, temperature, technique. The preferential choice of refrigerators on the basis of additional sub-attributes for the said attributes would be:
Price$\{e_1\} = \{$Low(e_{11}), High$(e_{12})\}$,
Type$\{e_2\} = \{$Single-door(e_{21}), Double-door$(e_{22})\}$
Temperature$\{e_3\}$, Cooling technique $\{e_4\} = \{$Direct$\{e_{41}$, Frost-free$\{e_{42}\}\}\}$ Let $E' = e_1 \times e_2 \times e_3 \times e_4$ be a set of sub-attributes express as $= \{(e_{11}, e_{21}, e_3, e_{41}),$
$(e_{11}, e_{21}, e_3, e_{42}), (e_{11}, e_{22}, e_3, e_{41}), (e_{11}, e_{22}, e_3, e_{42}), (e_{12}, e_{21}, e_3, e_{41}),$

$(e_{11}, e_{21}, e_3, e_{42}), (e_{11}, e_{22}, e_3, e_{41}), (e_{11}, e_{22}, e_3, e_{42})\}$.
For the calculation, all the sub-attributes can be posted as
$E' = \{e'_1, e'_2, e'_3, e'_4, e'_5, e'_6, e'_7, e'_8\}$.

The job is to classify the refrigerators on the basis of sub-attributes under consideration. Every refrigerator/product has to be given rating by some decision-maker based on the T-SFHSS sub-attributes. The summarized ratings in a comprehensive way have been computed and tabulated in Table 1.

Table 1: Decision-Experts rating value wrt each object

	e'_1	e'_2	e'_3	e'_4	e'_5	e'_6	e'_7	e'_8
X_1	(0.8,0.5,0.3)	(0.4,0.5,0.3)	(0.6,0.5,0.5)	(0.8,0.4,0.6)	(0.4,0.6,0.5)	(0.2,0.9,0.2)	(0.7,0.6,0.4)	(0.6,0.4,0.7)
X_2	(0.9,0.2,0.3)	(0.7,0.5,0.5)	(0.5,0.7,0.4)	(0.3,0.8,0.5)	(0.3,0.5,0.7)	(0.5,0.4,0.4)	(0.4,0.6,0.7)	(0.8,0.2,0.3)
X_3	(0.6,0.6,0.6)	(0.9,0.2,0.2)	(0.3,0.6,0.4)	(0.2,0.6,0.7)	(0.5,0.7,0.3)	(0.7,0.5,0.2)	(0.3,0.1,0.9)	(0.3,0.8,0.2)
X_4	(0.5,0.6,0.2)	(0.5,0.5,0.5)	(0.9,0.6,0.3)	(0.1,0.5,0.9)	(0.7,0.3,0.4)	(0.9,0.1,0.3)	(0.4,0.4,0.4)	(0.7,0.3,0.3)
X_5	(0.8,0.3,0.8)	(0.7,0.5,0.1)	(0.6,0.5,0.3)	(0.9,0.4,0.2)	(0.3,0.5,0.4)	(0.4,0.5,0.4)	(0.1,0.8,0.4)	(0.6,0.3,0.6)

Next, the measure S_h are computed for classification of the refrigerator/product X_j with the following:

Step (i): Computation of the similarity score for $S_h(X_j, X_k), (j, k = 1, 2, ..., 5)$. The desired matrix N would be

$$N = \begin{bmatrix} 1.000 & 0.712 & 0.589 & 0.648 & 0.836 \\ 0.712 & 1.000 & 0.675 & 0.677 & 0.693 \\ 0.589 & 0.675 & 1.000 & 0.664 & 0.617 \\ 0.648 & 0.677 & 0.664 & 1.000 & 0.604 \\ 0.836 & 0.693 & 0.617 & 0.604 & 1.000 \end{bmatrix}$$

Step (ii): Calculation of the $N^2 = N \circ N$ by using definition 10:

$$N^2 = \begin{bmatrix} 1.000 & 0.712 & 0.675 & 0.677 & 0.836 \\ 0.712 & 1.000 & 0.675 & 0.677 & 0.712 \\ 0.675 & 0.675 & 1.000 & 0.675 & 0.675 \\ 0.677 & 0.677 & 0.675 & 1.000 & 0.677 \\ 0.836 & 0.712 & 0.675 & 0.677 & 1.000 \end{bmatrix}$$

Since $N^2 \neq N$, Now we compute N^4 as $N^4 = N^2 \circ N^2$.

$$N^4 = \begin{bmatrix} 1.000 & 0.712 & 0.675 & 0.677 & 0.836 \\ 0.712 & 1.000 & 0.675 & 0.677 & 0.712 \\ 0.675 & 0.675 & 1.000 & 0.675 & 0.675 \\ 0.677 & 0.677 & 0.675 & 1.000 & 0.677 \\ 0.836 & 0.712 & 0.675 & 0.677 & 1.000 \end{bmatrix}$$

We get $N^4 = N^2$, therefore N^2 is identified to be a similar kind of matrix.

Step (iii): Suppose that the confidential level $\lambda = 0.712$. Further, computation of $\lambda-$cutting matrix N^λ using definition 12 has been done.

$$N^\lambda = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Step (iv): The proper classification of X_j into 3 different isometric classes with the help of N^λ matrix as, $\{X_1, X_2, X_5\}, \{X_3\}$ and $\{X_4\}$.

In view of the data in $\lambda-$cutting matrix, it is concluded that X_1 X_2 and X_5 refrigerator/product are found to be more identical with each other and other remaining ones are in one single class.

Table 2 illustrates the classification results obtained for different values of the confidence level λ. It is evident that, for a given confidence level, the decision-maker adopts a unique partitioning of the alternative set into a predefined number of classes. Moreover, the number of generated patterns increases proportionally with the value of λ.

Table 2: Clustering of X_j for different confidential levels

Confidential level	Clusters	Class
$0.000 \leq \lambda \leq 0.675$	$\{X_1, X_2, X_3, X_4, X_5\}$	1
$0.675 < \lambda \leq 0.677$	$\{X_1, X_2, X_4, X_5\}, X_3$	2
$0.677 < \lambda \leq 0.712$	$\{X_1, X_2, X_5\}, \{X_3\}, X_4$	3
$0.712 < \lambda \leq 0.836$	$\{X_1, X_5\}, \{X_2\}, \{X_3\}, \{X_4\}$	4
$0.836 < \lambda \leq 1.000$	$\{X_1\}, \{X_2\}, \{X_3\}, \{X_4\}, \{X_5\}$	5

In general, the decision-maker selects a confidence level $\lambda \in [0, 1]$. The corresponding clustering results and their associated λ-level matrices are presented as follows:

1. If $0.000 \leq \lambda \leq 0.675$,

$$N^\lambda = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

2. If $0.675 < \lambda \leq 0.677$,

$$N^\lambda = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$$

3. If $0.677 < \lambda \le 0.712$,

$$N^\lambda = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

4. If $0.712 < \lambda \le 0.836$,

$$N^\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

5. If $0.836 < \lambda \le 1.000$,

$$N^\lambda = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The λ-cut matrices indicate that the considered alternatives are partitioned into 1, 2, 3, 4, and 5 classes, respectively. It can be observed that, as the value of λ increases, the discrimination among the alternatives becomes more prominently visible.

5. Conclusion and Future Scope

Finally, in this study, we introduced novel distance and similarity measures for TSFHSS. The formulation of the proposed similarity measures has been presented in detail, and a numerical example has been provided to demonstrate the computation of similarity scores and their effectiveness. Several important theorems establishing the fundamental properties of the proposed measures have also been proved to ensure their mathematical validity. To illustrate their practical applicability, a clustering algorithm has been developed and implemented by classifying refrigerators based on selected characteristics, clearly demonstrating how the proposed measures facilitate reliable grouping of objects in uncertain environments. When ambiguity is represented in the TSFHSS framework, the proposed approach effectively handles situations where precise measurement tools for determining object similarity are unavailable. Beyond this illustrative example, the proposed TSFHSS-based measures have strong potential for broader applications in real-world problems involving uncertainty, such as medical diagnosis, image segmentation, pattern recognition, and multi-criteria decision-making. In future work, the study may be extended by developing more flexible variants of the proposed measures and applying them to other complex uncertain environments. The proposed framework can thus serve as a useful foundation for addressing a wide range of ambiguous and multi-attribute decision-making challenges.

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Conflicts of Interest

The authors declare no conflicts of interest.

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