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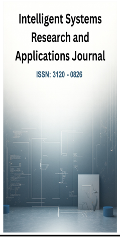

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Vertex Betweenness Centrality of Some Graph Classes with Applications

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ABSTRACT

Centrality is an excellent concept for measuring the important or critical vertices in graphs like social networks, biological networks, computer networks, etc. Every vertex may be an important one from a given perspective, depending on our assumptions or how we define importance. There are different centrality measures to characterize vertices in a network. Betweenness centrality is a crucial measure in network analysis based on all shortest paths between pairs of vertices. The betweenness centrality of a node point v is the sum of the ratios between the number of the shortest routes between each pair of other node points that pass through v and the total number of the shortest routes between them. This study presents new theoretical results on the measurement of betweenness centrality, relative betweenness centrality, and graph betweenness centrality for some special graph classes. In addition, we discuss real-life applications of the results studied in this article.

1. Introduction

In network analysis, one of the fundamental problems is to identify important vertices or edges in a large network. There are many centrality measurements for analyzing network problems. One of them is betweenness centrality (BC, in short). It is used to analyze social networks [1–4], biological networks [5–7], sexual networks and AIDS [8], computer networks [9], transportation networks [10], supply chain networks [11], terrorist networks [12, 13], organizational behavior [14], social psychology [15], temporal networks [16], complex networks [17–19], etc. Betweenness centrality measures the betweenness of a node (point/vertex) in a network based on shortest paths, assuming that information flows along the shortest paths. We can easily determine whether there is only one path between every pair of nodes.

If there are many paths between a pair of node points, determining betweenness centrality is

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complicated. A node with the highest betweenness centrality in a network can pass more information to other node points, indicating that it has more control over the network than others. Vertices having high betweenness centrality are frequently, though not invariably, effective choices for insertion in a dominating set [20]. BC has also an important role in decision-making problem [21].

Betweenness centrality of a node point v is symbolized by $C_B(v)$ and it is defined by $C_B(v) = \sum_{u \neq w \neq v} \frac{\sigma_{uw}(v)}{\sigma_{uw}}$, where σ_{uw} indicates the number of shortest routes between the node points u and w and $\sigma_{uw}(v)$ represents the number of shortest routes between u and w which pass through v. As the number of vertices in a network increases, betweenness centrality also increases. We cannot easily handle the large value. So, we normalize the value by dividing the maximum value of $C_B(v)$, which lies between 0 and 1. Freeman [22] proved that the central node point of a star graph with n node points gives the maximum value of $C_B(v)$, denoted it by $MaxC_B(v)$ for any graph with n node points, and that value is $\binom{n-1}{2} = \frac{(n-1)(n-2)}{2}$

Again, the *relative betweenness centrality* of a node point v is symbolized by $C'_B(v)$ and is defined by $C'_B(v) = \frac{C_B(v)}{MaxC_B(v)} = \frac{2C_B(v)}{(n-1)(n-2)}$, where $MaxC_B(v) = \frac{(n-1)(n-2)}{2}$.

Further, the betweenness centrality of a graph helps us to determine the tendency of a particular node point, which is more centric than other node points of the graph. Freeman [22] introduced the betweenness centrality of a graph as the average difference between the most central node and all other nodes in the graph. We use the notation $C_B(G)$ to represent the betweenness centrality of a graph G and it is defined as

$$C_B(G) = \frac{\sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]}{Max \sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]},$$ where $C_B(v^*)$ is the maximum value of $C_B(v_i)$, $v_i \in V(G)$ and $Max \sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]$ is the greatest possible sum of differences of betweenness centrality for any graph having n node points which equals to $(n-1)\binom{n-1}{2} = \frac{(n-1)^2(n-2)}{2}$ for star graphs [22], that is,

$$C_B(G) = \frac{2\sum_{i=1}^{n}[C_B(v^*) - C_B(v_i)]}{(n-1)^2(n-2)}.$$

or, $$C_B(G) = \frac{\sum_{i=1}^{n}[C'_B(v^*) - C'_B(v_i)]}{(n-1)}.$$

1.1 *Review of the Related Works*

In 1948, Bavelas [23] first introduced the concept of centrality and applied it in human communication. In 1954, Shaw [24] studied betweenness centrality, but he did not provide a method to measure it. Freeman [22] first introduced the mathematical expression of betweenness centrality in 1977. He also defined the relative betweenness centrality of a vertex and the betweenness centrality of the graph and proved that the central vertex of the star graph gives the maximum betweenness centrality among all graphs with n vertices. In 1978, Freeman [25] defined the graph centrality for social networks. After that, Brandes [26] designed an $O(mn)$ time faster algorithm for betweenness centrality in 2001. He [27] also implemented necessary software tools for analyzing the network by betweenness centrality in 2008. In the same year, Kintali [28] presented a randomized parallel algorithm and an algebraic method to determine the betweenness centrality of all nodes in the network. Estrada et al. introduced the subgraph centrality for complex networks [29]. They also defined communicability centrality by the exponential of the adjacency matrix and also showed that it is related to the Frechet derivative in 2009 [30]. In 2012, Puzis et al. [31] introduced heuristics to speed up the computation of betweenness centrality. Thereafter, Gago et al. [32] studied the betweenness centrality of uniform graphs in 2013. The following year, Zaoli et al. [33] defined the betweenness centrality of temporal multiplexes for topological and temporal structures. In the same year, Unnithan et al. [34] studied the betweenness centrality of some basic graphs. In 2015, Suppa et al. [35] determined betweenness centrality in social networks using a clustering approach. In 2018, Bergamini et al. [36] presented a new dynamic algorithm to find the betweenness centrality of a vertex after adding a few edges to it.

Recently, several studies have focused on the challenge of dynamic updates; for instance, Satotani et al. [37] introduced an efficient method for recalculating BC following edge deletion, while Xiang et al. [38] utilized shortest-path approximations to streamline the estimation process. The scope of our research topic has also expanded to include temporal and evolving networks, as seen in the work by Zhang et al. [39] and Naima [40] on shortest walk variants. The practical utility of these measures is equally significant. Wang et al. [41] demonstrated how BC distributions can model the spread of rumors, and Nandi et al. [42] applied these metrics to enhance the resilience of transport infrastructure against disruptions. Furthermore, theoretical foundations continue to be refined through investigations into the Cartesian product of graphs [43] and the intrinsic link between BC and shortest-path distributions [44]. Collectively, these advancements underscore the critical role of BC in modern network science.

1.2 *Result*

This paper studies the theoretical development for computing the betweenness centrality of several special graph classes, including the friendship graph, fan graph, lollipop graph, umbrella graph, n-net graph, n-sunlet graph, and helm graph. We also study the relative betweenness centrality and graph betweenness centrality of these graphs. The practical implications of our findings are illustrated through an analysis of a real-life network application.

1.3 *Arrangement of the Article*

In the next section, we state and prove a theorem about the betweenness centrality of all vertices in the friendship graph. We also study the relationship between relative betweenness centrality and graph betweenness centrality in the same section. In Section 3, we find the betweenness centrality and the relative betweenness centrality of the fan graph. Section 4 describes the same centrality measurements of the lollipop graph. In Sections 5 and 6, we study the considered problem for the umbrella and n-net graphs, respectively. We discuss the same centrality measurements of the n-sunlet graph in Section 7. Section 8 presents betweenness centrality, relative betweenness centrality, and graph betweenness centrality of the helm graph. Section 9 presents real-life applications of the studied results. In the final section, we write the paper's conclusion.

2. Betweenness Centrality of Friendship Graphs

A friendship graph F_n is an undirected planar graph. If we join n copies of the cycle C_3 with a common node point, then it becomes a friendship graph. The cardinalities of the vertex set and the edge set of F_n are, respectively, $2n + 1$ and $3n$. A friendship graph F_4 is shown in Figure 1.

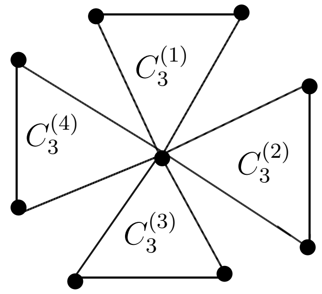

Figure 1: Friendship graph F_4

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Theorem 1 *The betweenness centrality of any node point v of F_n is given by*

$$C_B(v) = \begin{cases} 2n(n-1), & \text{if } v \text{ is the central node point of } F_n, \\ 0, & \text{otherwise.} \end{cases}$$

.

Proof. Let $C_3^{(1)}, C_3^{(2)}, \cdots, C_3^{(n)}$ be the n cycles of length three form the graph F_n, u is the central node point of F_n. Each cycle consists of 2 vertices except u. Therefore, F_n contains $2n$ vertices except u. If v is any vertex of F_n except u, then the shortest routes/paths between any pairs of non-adjacent vertices (except v) of F_n must pass through the common vertex (i.e., the central vertex) and never pass through v. Therefore, the betweenness centrality of any vertex of F_n except the common vertex is 0.

Let v be the common vertex of F_n. As each cycle consists of 2 vertices except the common vertex, the total number of vertices of F_n except the vertex of the cycle $C_3^{(1)}$ and the vertex v is $2n-2$. The shortest route between any vertex of $C_3^{(1)}$ and any one of the vertices of $C_3^{(2)}, C_3^{(3)}, \cdots$ and $C_3^{(n)}$ passes through v. So, each vertex of $C_3^{(1)}$ except v contributes centrality $2n-2$. As, $C_3^{(1)}$ consist 2 vertices except v, therefore the vertices of $C_3^{(1)}$ contribute centrality $2(2n-2)$ to v. Again, the shortest route/path between any vertex of $C_3^{(2)}$ and any one of the vertices of $C_3^{(3)}, C_3^{(4)}, \cdots, C_3^{(n-1)}$ and $C_3^{(n)}$ passes through v. Further, the shortest route whose one end vertex is in $V(C_3^{(1)})$ and the other end vertex is in $V(C_3^{(2)})$ has already been counted. Thus, the total number of vertices of the cycles $C_3^{(3)}, C_3^{(4)}, \cdots, C_3^{(n)}$ is $2n-4$. The vertices of $C_3^{(2)}$ except v contribute centrality $2(2n-4)$ to v. Similarly, the vertices of $C_3^{(3)}, C_3^{(4)}, \cdots, C_3^{(n)}$ contribute centrality $2(2n-6), 2(2n-8), \cdots, 2(2), 0$, respectively.

Therefore the $C_B(v)$ is

$$\begin{aligned} &2(2n-2) + 2(2n-4) + \cdots + 2(2) \\ &= 2\{(2n-2) + (2n-4) + \cdots + 2\} \\ &= 2 \times 2\{(n-1) + (n-2) + \cdots + 1\} \\ &= \frac{4(n-1)(n-1+1)}{2} \\ &= 2n(n-1). \end{aligned}$$

□

Note 1: The relative betweenness centrality of any node point v is given by $C'_B(v) = \dfrac{C_B(v)}{MaxC_B(v)}$.

Here, $MaxC_B(v) = \dfrac{(2n+1-1)(2n+1-2)}{2} = \dfrac{2n(2n-1)}{2}$ (Using the result for the star graph with $2n+1$ vertices).

Therefore, we can write $C'_B(v) = \dfrac{2C_B(v)}{2n(2n-1)}$

$$= \begin{cases} \dfrac{2n-2}{2n-1}, & \text{if } v \text{ is the central node point of } F_n, \\ \\ 0, & \text{otherwise.} \end{cases}$$

Note 2: From the above result, we see that $C'_B(v^*) = \dfrac{2n-2}{2n-1}$.

So, the graph betweenness centrality $C_B(F_n) = \dfrac{\sum_{i=1}^{2n+1}[C'_B(v^*) - C'_B(v_i)]}{(2n+1-1)}$

$$= \frac{\left[\sum_{i=1}^{2n}\left(\frac{2n-2}{2n-1} - 0\right)\right] + 0}{2n}$$

$$= \frac{2n\left(\frac{2n-2}{2n-1}\right)}{2n}$$

$$= \frac{2n-2}{2n-1}$$

3. Betweenness Centrality of Fan Graphs

The fan graph $F(m,n)$ is obtained by the graph joining of two graphs—one is the null graph $\overline{K_m}$ of m vertices, and another is the path graph P_n of n vertices, i.e., $F(m,n) = \overline{K_m} + P_n$ where $n > 1$. Let $\{u_1, u_2, \cdots, u_m\}$ be the set of m node points/vertices of $\overline{K_m}$ and $\{v_1, v_2, \cdots, v_n\}$ be the set of n node points of P_n. The $F(m,n)$ has $m+n$ nodes and $mn+n-1$ edges. If $m=1$, we call it the usual fan graph; if $m=2$, the fan graph is called the double fan graph. Figure 2 shows a fan graph $F(3,4)$.

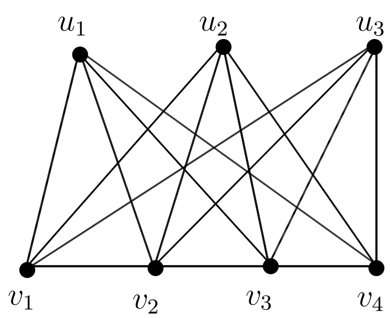

Figure 2: Fan graph $F(3,4)$

.

Theorem 2 *The $C_B(v)$ of any node point v of $F(m,n)$ is given by*

$$C_B(v) = \begin{cases} \frac{n-2}{m+1} + \frac{(n-3)(n-2)}{2m}, & \text{if } v \in \overline{K_m}, \\ \frac{1}{n} \times \binom{m}{2}, & \text{if } m > 1 \text{ and } v \text{ is an extremity of } P_n, \\ \frac{1}{n} \times \binom{m}{2} + \frac{1}{m+1}, & \text{if } m > 1 \text{ and } v \text{ is an intermediate vertex of } P_n. \end{cases}$$

.

Proof. Suppose that $\{u_1, u_2, \cdots, u_m, v_1, v_2, \cdots, v_n\}$ is the set of node points of $F(m,n) = \overline{K_m} + P_n, m > 1$, where $V(\overline{K_m}) = \{u_1, u_2, \cdots, u_m\}$ and $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. The length of the shortest route between each pair of vertices of $F(m,n)$ is 1 or 2. Let v be any node point of $\overline{K_m}$. So, adjacent node points do not contribute to $C_B(v)$. First, we calculate the contribution of a pair of non-adjacent node points (v_1 and v_3) to $C_B(v)$. Obviously $d(v_1, v_3) = 2$. The number of shortest routes between v_1 and v_3 is $m+1$, and among these only one route between v_1 and v_3 passes through v. So, the pair (v_1, v_3) contributes $\frac{1}{m+1}$ to $C_B(v)$. Also, there are m shortest routes between v_1 and any one of $\{v_4, v_5, ..., v_n\}$, among which only one shortest route passes through v. The total number of pairs between v_1 and any two of them $\{v_4, v_5, \cdots, v_n\}$ is $n-3$. Therefore, v_1 gives value to the $C_B(v)$ is $\dfrac{1}{m+1} + \dfrac{n-3}{m}$. Similarly, $v_2, v_3, \cdots, v_{n-3}, v_{n-2}$ contribute the values to the $C_B(v)$ area, respectively, $\dfrac{1}{m+1} + \dfrac{n-4}{m}, \dfrac{1}{m+1} + \dfrac{n-5}{m}, \cdots, \dfrac{1}{m+1} + \dfrac{1}{m}, \dfrac{1}{m+1}$. In addition, these other pairs each contribute 0 to $C_B(v)$. Therefore, the value of $C_B(v)$ is $(\dfrac{1}{m+1} + \dfrac{n-3}{m}) + (\dfrac{1}{m+1} + \dfrac{n-4}{m}) + \cdots + (\dfrac{1}{m+1} + \dfrac{1}{m}) + \dfrac{1}{m+1}$

$= \dfrac{n-2}{m+1} + \dfrac{1}{m} \times \{(n-3) + (n-4) + \cdots + 1\}$

$= \dfrac{n-2}{m+1} + \dfrac{1}{m} \times \dfrac{(n-3)(n-2)}{2}$

$= \dfrac{n-2}{m+1} + \dfrac{(n-3)(n-2)}{2m}$.

If $v = v_1$, where $v_1 \in P_n$, then the shortest routes between any pair of vertices of P_n do not pass through v. The vertices of P_n give the value 0 to $C_B(v)$ and any pair of vertices of $\overline{K_m}$ contributes the value $\frac{1}{n}$ to $C_B(v)$. The number of pairs of vertices of $\overline{K_m}$ is $\binom{m}{2}$. Therefore, $C_B(v) = \dfrac{1}{n} \times \binom{m}{2}$. Similarly, if $v = v_n$ then $C_B(v) = \dfrac{1}{n} \times \binom{m}{2}$.

If v is any vertex of P_n except v_1 and v_n then any pairs of vertices of $\overline{K_m}$ gives the value $\dfrac{1}{n}$ to $C_B(v)$ and there are $\binom{m}{2}$ distinct pairs of node points in $\overline{K_m}$. So, all pairs of node points of $\overline{K_m}$ contribute $\frac{1}{n} \times \binom{m}{2}$ to $C_B(v)$. Again, only a pair of vertices (which are adjacent to v) of P_n contribute the value $\dfrac{1}{m+1}$ to $C_B(v)$. All other pairs of vertices of P_n contribute 0 to $C_B(v)$.

Therefore, $C_B(v) = \dfrac{1}{m+1} + \dfrac{1}{n} \times \binom{m}{2}$. □

Note 3: If $m = 1$, then the betweenness centrality of v_1 and v_n are 0, the betweenness centrality of each of $v_2, v_3, \cdots, v_{n-1}$ is $\dfrac{1}{2}$ and for the central vertex u_1, $C_B(u_1) = \dfrac{(n-2)^2}{2}$.

Note 4: The relative betweenness centrality is calculated in the following:

$$C'_B(v) = \frac{C_B(v)}{MaxC_B(v)}$$

Here, $MaxC_B(v) = \dfrac{(m+n-1)(m+n-2)}{2}$ (Using the result for the star graph with $m+n$ vertices).

So, $C'_B(v) = \dfrac{2C_B(v)}{(m+n-1)(m+n-2)}$

$$= \begin{cases} \frac{2}{(m+n-1)(m+n-2)}\{\frac{n-2}{m+1} + \frac{(n-3)(n-2)}{2m}\}, & \text{if } v \in \overline{K_m}, \\ \frac{2}{(m+n-1)(m+n-2)}\{\frac{1}{n} \times \binom{m}{2}\}, & \text{if } m > 1 \text{ and } v \text{ is an extremitiy of } P_n, \\ \frac{2}{(m+n-1)(m+n-2)}\{\frac{1}{n} \times \binom{m}{2} + \frac{1}{m+1}\}, & \text{if } m > 1 \text{ and } v \text{ is an intermediate vertex of } P_n. \end{cases}$$

4. Betweenness Centrality of Lollipop Graphs

In graph theory, a lollipop graph is a special kind of simple, connected graph. It is a geodetic graph [45]. The (m, n) lollipop graph, symbolized by $L_{m,n}$, consists of a K_m (complete graph of m node points) and a P_n (path graph of n node points), linked by a bridge.Obviously $|V(L_{m,n})| = m + n$ and $|E(L_{m,n})| = \binom{m}{2} + n$. Let $V(K_m) = \{v_1, v_2, \cdots, v_m\}$ and $V(P_n) = \{v_{m+1}, v_{m+2}, \cdots, v_{m+n}\}$. Here we consider that v_m is the one end vertex of the linking bridge that lies on K_m and v_{m+1} is the other end vertex of the same bridge that lies on P_n. A lollipop graph $L_{6,4}$ is shown in Figure 3.

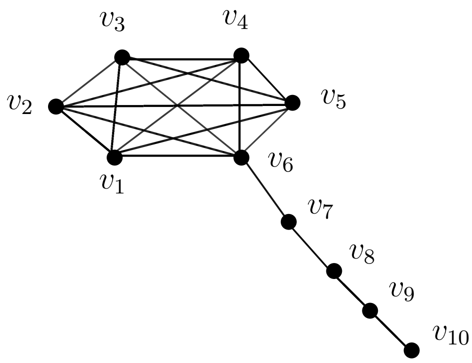

Figure 3: Lollipop graph $L_{6,4}$

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Theorem 3 *The $C_B(v)$ of any node point v of $L_{m,n}$ is given by*

$$C_B(v) = \begin{cases} 0, & \text{if } v \in K_m, v \neq v_m, \\ n(m-1), & \text{if } v \in K_m, v = v_m, \\ (n-p)(m+p-1), & \text{if } v \in P_n, v = v_{m+p}, p = 1, 2, \cdots, n. \end{cases}$$

.

Proof. Let $L_{m,n}$ be a lollipop graph and $V(L_{m,n}) = \{v_1, v_2, \cdots, v_m, v_{m+1}, v_{m+2}, \cdots, v_{m+n}\}$, where $\{v_1, v_2, \cdots, v_m\}$ is the set of node points of the K_m and $\{v_m, v_{m+1}, v_{m+2}, \cdots, v_{m+n}\}$ is the set of node points of the P_n. Also, we assume that v is any node point of K_m except v_m. As any two vertices of K_m are adjacent, the shortest route between any vertices of K_m except v and any vertices of P_n never passes through the node v. Therefore, $C_B(v) = 0$.

If $v \in K_m$ and $v = v_m$, then the shortest route between each pair of vertices, among which one lies in $V(K_m) - \{v\}$ and the other lies in $V(P_n)$ must pass through the node v. There are $n(m-1)$ such pairs of vertices, and each pair contributes the value 1 to $C_B(v)$. Therefore, $C_B(v) = n(m-1)$.

Again, let v be any vertex of P_n. Therefore, $v = v_{m+p}$ where $p = 1, 2, \cdots, n$. It is obvious that there exist only one shortest path between any one of $\{v_1, v_2, \cdots, v_m, v_{m+1}, v_{m+2}, \cdots, v_{m+p-1}\}$ and any one of $\{v_{m+p+1}, v_{m+p+2}, \cdots, v_{m+n}\}$ that must pass through v. The number of such pairs is $(m+p-1)(m+n-\overline{m+p+1}+1) = (m+p-1)(n-p)$ each pair contributes the value 1 to $C_B(v)$. Therefore, $C_B(v) = 1.(n-p)(m+p-1) = (n-p)(m+p-1)$. □

Note 5: The relative betweenness centrality of any node point v is $C'_B(v) = \dfrac{C_B(v)}{MaxC_B(v)}$.

In this case, $MaxC_B(v) = \dfrac{(m+n-1)(m+n-2)}{2}$ using the result for the star graph with $m+n$ vertices.

$$\therefore C'_B(v) = \frac{2C_B(v)}{(m+n-1)(m+n-2)}$$

$$= \begin{cases} 0, & \text{if } v \in K_m, v \neq v_m, \\ \dfrac{2n(m-1)}{(m+n-1)(m+n-2)}, & \text{if } v \in K_m, v = v_m, \\ \dfrac{2(n-p)(m+p-1)}{(m+n-1)(m+n-2)}, & \text{if } v \in P_n, v = v_{m+p}, p = 1, 2, \cdots, n. \end{cases}$$

Note 6: From the above result, we observe that $C_B(v^*) = n(m-1)$.

So, the graph betweenness centrality $C_B(L_{m,n}) = \dfrac{2\sum_{i=1}^{m+n}\Big[C_B(v^*) - C_B(v_i)\Big]}{(m+n-1)^2(m+n-2)}$

$$= \frac{2\sum_{i=1}^{m-1}\Big[C_B(v^*) - C_B(v_i)\Big] + 2\Big[C_B(v^*) - C_B(v_m)\Big] + 2\sum_{i=m+1}^{m+n}\Big[C_B(v^*) - C_B(v_i)\Big]}{(m+n-1)^2(m+n-2)}$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Bigg[\sum_{i=1}^{m-1}\{n(m-1)-0\} + 0 + \{n(m-1) - m(n-1)\} + \{n(m-1) - (m+1)(n-2)\} + \cdots + \{n(m-1) - 2(m+n-3)\} + \{n(m-1) - (m+n-2)\} + \{n(m-1) - 0\}\Bigg]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Bigg[(m-1)n(m-1) + \sum_{i=1}^{n-1} n(m-1) - \{m(n-1) + (m+1)(n-2) + \cdots + 2(m+n-3) + (m+n-2)\} + n(m-1)\Bigg]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[n(m-1)(m-1+1) + (n-1)n(m-1) - \{m(n-1) + m(n-2) + \cdots + m.2 + m.1\}$$

$$- \{0 + (n-2).1 + (n-3).2 + \cdots + 2.(n-3) + 1.(n-2)\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[nm(m-1) + n(n-1)(m-1) - \{m.\frac{n(n-1)}{2}\}$$
$$- \{\frac{(n-2)^3 - 3(n-2)^2 + 2(n-2)}{6}\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[nm(m-1) + n(n-1)(m-1-\frac{m}{2})$$
$$- \frac{n^3 - 6n^2 + 12n - 8 - 3n^2 + 12n - 12 + 2n - 4}{6}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[\frac{6nm^2 - 6nm + 3mn^2 - 6n^2 - 3nm + 6n - n^3 + 9n^2 - 26n + 24}{6}\Big]$$

$$= \frac{1}{(m+n-1)^2(m+n-2)}\Big[\frac{6nm^2 - 9nm + 3mn^2 + 3n^2 - n^3 - 20n + 24}{3}\Big]$$

5. Betweenness Centrality of Umbrella Graphs

An umbrella graph $U(m,n)$ is formed by merging one end vertex of a path graph P_n and the central node point of a fan graph $F(1,m)$. A $U(m,n)$ has $m+n$ node points and $2m+n-2$ edges. Let $u_1, u_2, \cdots, u_n$ be the node points of P_n and $u_n, u_{n+1}, u_{n+2}, \cdots, u_{n+m}$ be the node points of $F(1,m)$, where u_n is the central node point of $F(1,m)$ as well as an end node point of P_n. An umbrella graph $U(5,4)$ is shown in Figure 4.

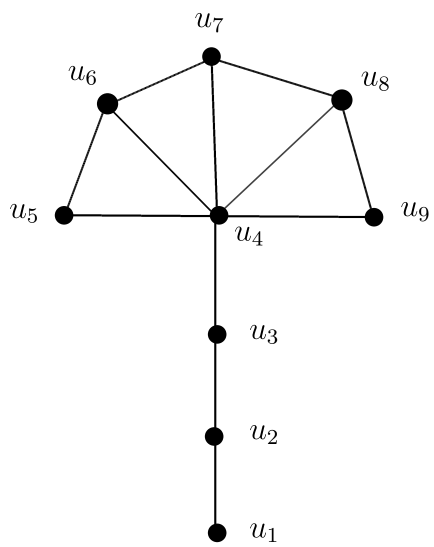

Figure 4: Umbrella graph $U(5,4)$

.

Theorem 4 *The $C_B(v)$ of any node point v of $U(m,n)$ is given by*

$$C_B(v) = \begin{cases} 0, & \textit{if } v \in F(1,m), v = u_{n+1} \textit{ or } v = u_{m+n}, \\ \frac{1}{2}, & \textit{if } v \in F(1,m), v \neq u_{n+1} \textit{ and } v \neq u_{m+n}, \\ m(n-1) + \frac{(m-2)^2}{2}, & \textit{if } v = u_n, \\ (m+p)(n-p-1), & \textit{if } v \in P_n, v = u_{n-p}, p = 1, 2, \cdots, n-1. \end{cases}$$

.

Proof. Let $U(m,n)$ be a umbrella graph and $\{u_1, u_2, \cdots, u_n, u_{n+1}, u_{n+2}, \cdots, u_{m+n}\}$ be the set of node points of $U(m,n)$, where $V(P_n) = \{u_1, u_2, \cdots, u_n\}$, $V(F(1,m)) = \{u_n, u_{n+1}, u_{n+2}, \cdots, u_{m+n}\}$ and u_n is the central node point of $F(1,m)$.
Let us consider, $v = u_{n+1} \in F(1,m)$. As the central vertex u_n of $F(1,m)$ is directly connected to all other vertices of $F(1,m)$, so, the shortest route between a vertex (except v) of $F(1,m)$ and a vertex of P_n never passes through v. Again, the shortest route between each pair of vertices (except $v = u_{n+1}$) of $F(1,m)$ as well as P_n does not pass through v. So, $C_B(v) = 0$. Again, if $v = u_{n+m}$, then we can similarly show that $C_B(v = u_{n+m}) = 0$.

If v is any vertex of $F(1,m)$ except u_{n+1} and u_{n+m}, then the shortest route between a vertex of $F(1,m)$ and a vertex of P_n and the shortest route between each pair of vertices of P_n does not pass through v. Also, the shortest route between the pair of node points/vertices (both are adjacent to v) of $F(1,m)$ passes through v and that pair of vertices of $F(1,m)$ gives the value $\frac{1}{2}$ to $C_B(v)$. So, in this case $C_B(v) = \frac{1}{2}$.

Let $v = u_{n-p}$ be any vertex of P_n except u_n, where $p = 1, 2, \cdots, n-1$. Now, There is only one shortest path between a vertex of $\{u_1, u_2, \cdots, u_{n-p-1}\}$ and a vertex of $\{u_{n-p+1}, u_{n-p+2}, \cdots, u_n, u_{n+1}, \cdots, u_{n+m}\}$ must passes through v, and this type pair of vertices contributes the value 1 to $C_B(v)$. The number of such pairs is $(n-p-1)(m+n-\overline{n-p+1}+1) = (n-p-1)(m+p)$. Therefore, $C_B(v) = 1.(n-p-1)(m+p) = (n-p-1)(m+p)$.

If $v \in P_n$ and $v = u_n$ then the shortest path between a vertex of $\{u_1, u_2, \cdots, u_{n-1}\}$ and a vertex of $\{u_{n+1}, u_{n+2}, \cdots, u_{n+m}\}$ must passes through v, and this type pair of vertices contributes the value 1 to $C_B(v)$. Also, the number of such pairs of vertices is $(n-1)(m+n-\overline{n+1}+1) = m(n-1)$. This pair of vertices contributes the value $m(n-1)$ to $C_B(v)$. Besides these, all pairs of vertices of $F(1,m)$, excluding u_n give the value $\frac{(m-2)^2}{2}$ (see theorem 2) to $C_B(v)$. Therefore, $C_B(v)$ is $m(n-1) + \frac{(m-2)^2}{2}$. □

Note 7: The relative betweenness centrality $C'_B(v) = \frac{C_B(v)}{MaxC_B(v)}$.
Here, $U(m,n)$, $MaxC_B(v) = \frac{(m+n-1)(m+n-2)}{2}$ (Using the result for the star graph with $m+n$ vertices).
Therefore, $C'_B(v) = \frac{2C_B(v)}{(m+n-1)(m+n-2)}$

$$= \begin{cases} 0, & \text{if } v \in F(1,m), v = u_{n+1} \text{ or } v = u_{m+n}\,, \\ \dfrac{1}{(m+n-1)(m+n-2)}, & \text{if } v \in F(1,m), v \neq u_{n+1} \text{ and } v \neq u_{m+n}, \\ \dfrac{2}{(m+n-1)(m+n-2)}\{m(n-1)+\dfrac{(m-2)^2}{2}\}, & \text{if } v \in P_n \text{ and } v = u_n, \\ \dfrac{2(m+p)(n-p-1)}{(m+n-1)(m+n-2)}, & \text{if } v \in P_n, v = u_{n-p}, p = 1,2,\cdots,n. \end{cases}$$

Note 8: From the above result, we see that $C_B(v^*) = m(n-1) + \dfrac{(m-2)^2}{2}$.

So, $C_B(U_{m,n}) = \dfrac{2\sum\limits_{i=1}^{m+n}\Big[C_B(v^*) - C_B(v_i)\Big]}{(m+n-1)^2(m+n-2)}$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[\sum_{i=1}^{n}\{C_B(v^*)-C_B(v_i)\} + \sum_{i=n+1}^{n+m}\{C_B(v^*)-C_B(v_i)\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[2.\{m(n-1)+\tfrac{(m-2)^2}{2}-0\} + (m-2)\{m(n-1)+\frac{(m-2)^2}{2}-\frac{1}{2}\}$$
$$+0+1.\{m(n-1)+\frac{(m-2)^2}{2}-0\} + \{m(n-1)+\tfrac{(m-2)^2}{2}$$
$$-(m+1)(n-2)\} + \{m(n-1)+\frac{(m-2)^2}{2}-(m+2)(n-3)\}$$
$$+\cdots+\{m(n-1)+\frac{(m-2)^2}{2}-(m+n-2)\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[(m+n-1)\{m(n-1)+\frac{(m-2)^2}{2}\} - \frac{m-2}{2} - \{m(n-2)$$
$$+m(n-3)+\cdots+m.2+m.1\} - \{0+(n-2).1+(n-3).2$$
$$+\cdots+2.(n-3)+1.(n-2)\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[(m+n-1)\{m(n-1)+\frac{(m-2)^2}{2}\} - \frac{m-2}{2} - m\{(n-2)+(n-3)$$
$$+\cdots+1\} - \{1.(n-2)+2.(n-3)+\cdots+(n-2).1\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[(m+n-1)\{m(n-1)+\frac{(m-2)^2}{2}\} - \frac{m-2}{2}$$
$$-\frac{m(n-2)(n-1)}{2} - \{\frac{(n-2)^3-3(n-2)^2+2(n-2)}{6}\}\Big]$$

$$= \frac{2}{(m+n-1)^2(m+n-2)}\Big[(nm^2+mn^2-mn-m^2-mn+m)$$
$$+\frac{m^3-4m^2+4m+nm^2-4mn+4n-m^2+4m-4}{2} - \frac{m-2}{2}$$
$$-\frac{mn^2-3mn+2m}{2} - \frac{n^3-6n^2+12n-8-3n^2+12n-12+2n-4}{6}\Big]$$

$$= \frac{2(6nm^2+6mn^2-6mn-6m^2-6mn+6m+3m^3-12m^2+12m+3nm^2-12mn)}{6(m+n-1)^2(m+n-2)}$$

$$+\frac{2(12n-3m^2+12m-12-3m+6-3mn^2+9mn-6m-n^3+9n^2-26n+24)}{6(m+n-1)^2(m+n-2)}$$
$$=\frac{3m^3-n^3+9nm^2+3mn^2-15mn-21m^2-9n^2+21m-14n+18}{3(m+n-1)^2(m+n-2)}$$

6. Betweenness Centrality of n-Net Graphs

The n-net graph G is obtained by joining the vertices of $\overline{K_3}$ (a null graph having three node points) with the central node point and two end vertices of the fan graph $F(1,n)$ by three bridges. The n-net graph G has $n+4$ node points and $2(n+1)$ edges. Let $\{v_1, v_2, v_3\}$ be the vertex set of $\overline{K_3}$ and $\{u_1, u_2, ..., u_n, u\}$ be the vertex set of $F(1,n)$, where u is the central vertex and u_1, u_n are the end vertices of $F(1,n)$. A 6-net graph is shown in Figure 5.

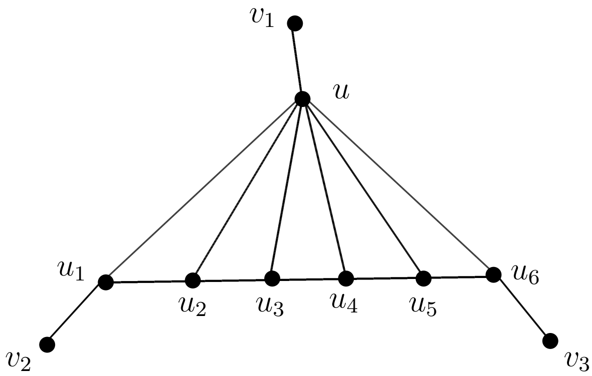

Figure 5: 6-net graph

.

Theorem 5 *The $C_B(v)$ of any node point v of n-net graph is given by*

$$C_B(v)=\begin{cases}0, & \text{if } v \text{ is a pendant vertex},\\ n+2, & \text{if } n>3, v=u_1 \text{ and } v=u_n\,,\\ 1, & \text{if } n>3, v=u_2 \text{ and } v=u_{n-1},\\ \frac{1}{2}, & \text{if } v=u_i \text{ and } i=3,4,\cdots,n-2,\\ \frac{n(n+2)}{2}, & \text{if } n>3, v=u.\end{cases}$$

.

Proof. Suppose that the n-net graph G has the node points $u_1, u_2, \cdots, u_n, u, v_1, v_2, v_3$, among which v_1, v_2, v_3 are the node points of $\overline{K_3}$ and u is the central vertex of $F(1,n)$.

Let v be any vertex of $\overline{K_3}$. i.e., v is a pendant vertex of the n-net graph G. No shortest route between any pair of node points of G passes through v. So, $C_B(v)=0$.

Again, let $v \in G$ and $v = u_i, i = 3, 4, \cdots, n-2$, $n > 3$. The only pair of vertices of $F(1,n)$ containing two adjacent vertices of v except the central vertex u of $F(1,n)$ contributes $\frac{1}{2}$ to $C_B(v)$. Therefore, $C_B(v) = \dfrac{1}{2}$.

If $v \in G$ and $v = u_2$ then each pair of vertices among (u_1, u_3) and (v_2, u_3) gives $\dfrac{1}{2}$ to $C_B(v)$. Therefore the betweennesss centrality of v is $2.\dfrac{1}{2} = 1$. Similarly, if $v = u_{n-1}$ then $C_B(v) = 1$.

If $v = u_1$, $n > 3$ then there exist only one shortest path between the vertex v_2 and a vertex of $\{v_1, v_3, u, u_2, u_4, u_5, \cdots, u_n\}$ and these pairs of vertices give the value $1.(n+2)$ to $C_B(v)$. Therefore, $C_B(v) = (n+2)$. Similarly, if $v = u_n$, then $C_B(v) = n+2$.

If $v = u$ and $n > 3$, then the shortest path between the vertex v_1 and a vertex of the set $\{u_1, u_2, \cdots, u_n, v_2, v_3\}$ pass through v and the number of such pairs of vertices is $(n+2)$. So, these pairs contribute $n+2$ to $C_B(v)$. Again, vertices of fan graph $F(1,n)$ gives the value $\dfrac{(n-2)^2}{2}$ (by theorem 2) to $C_B(v)$. Also, the only shortest path between the vertex v_2 and a vertex of $\{u_4, u_5, \cdots, u_n\}$ passes through v and the number of such pairs of vertices is $(n-3)$. These pairs give the value $n-3$ to $C_B(v)$. Further, there are two shortest paths between v_2 and u_3, among which one passes through v. So, this pair contributes $\dfrac{1}{2}$ to $C_B(v)$. Similarly, the only shortest path between the vertex v_3 and a vertex of $\{u_1, u_2, \cdots, u_{n-3}\}$ passes through v and the number of such pairs of vertices is $(n-3)$. These pairs give the value $n-3$ to $C_B(v)$. Further, there are two shortest paths between v_3 and u_{n-2}, among which one is passes through v. So, this pair contributes $\dfrac{1}{2}$ to $C_B(v)$. Also, the pair (v_2, v_3) contributes 1 to $C_B(v)$. Therefore, $C_B(v) = (n+2) + \dfrac{(n-2)^2}{2} + 2\{(n-3) + \dfrac{1}{2}\} + 1$

$$= \frac{2n + 4 + n^2 - 4n + 4 + 4n - 12 + 2 + 2}{2}$$
$$= \frac{n^2 + 2n}{2}$$
$$= \frac{n(n+2)}{2}.$$

Hence, the result is proved. □

Note 9: If $n = 3$ then $C_B(u_1) = C_B(u_3) = C_B(u) = n+2$ and $C_B(u_2) = 2$.
Note 10: If $n = 2$ then $C_B(u_1) = C_B(u_2) = C_B(u) = n+2$.
Note 11: The relative betweenness centrality is $C'_B(v) = \frac{C_B(v)}{MaxC_B(v)}$.

Here, $MaxC_B(v) = \dfrac{(n+4-1)(n+4-2)}{2} = \dfrac{(n+3)(n+2)}{2}$ (Using the result for the star graph with $n+4$ vertices).

Therefore, $C'_B(v) = \dfrac{2C_B(v)}{(n+3)(n+2)}$

$$= \begin{cases} 0, & \text{if } v \text{ is a pendant vertex,} \\ \dfrac{2}{n+3}, & \text{if } n > 3, v = u_1 \text{ and } v = u_n \text{ ,} \\ \dfrac{2}{(n+3)(n+2)}, & \text{if } n > 3, v = u_2 \text{ and } v = u_{n-1}, \\ \dfrac{1}{(n+3)(n+2)}, & \text{if } v = u_i \text{ and } i = 3, 4, \cdots, n-2, \\ \dfrac{n}{n+3}, & \text{if } n > 3, v = u. \end{cases}$$

Note 12: For the n-net graph G, $C_B(v^*) = \dfrac{n(n+2)}{2}$.

Therefore, $C_B(G) = \dfrac{2\sum\limits_{i=1}^{n+4}\Big[C_B(v^*) - C_B(v_i)\Big]}{(n+4-1)^2(n+4-2)}$

$$= \frac{2}{(n+3)^2(n+2)}\Big[3\{\frac{n(n+2)}{2} - 0\} + 2\{\frac{n(n+2)}{2} - (n+2)\} + 2\{\frac{n(n+2)}{2} - 1\} + 0 + (n-4)\{\frac{n(n+2)}{2} - \frac{1}{2}\}\Big]$$

$$= \frac{2}{(n+3)^2(n+2)}\Big[\frac{3n(n+2)}{2} + 2(n+2)(\frac{n}{2} - 1) + \frac{2(n^2+2n-2)}{2} + \frac{(n-4)(n^2+2n-1)}{2}\Big]$$

$$= \frac{2}{(n+3)^2(n+2)}\Big[\frac{3n(n+2)}{2} + 2(n+2)\frac{(n-2)}{2} + \frac{2n^2+4n-4}{2} + \frac{n^3+2n^2-n-4n^2-8n+4}{2}\Big]$$

$$= \frac{3n^2+6n+2n^2+4n-4+2n^2-8+n^3+2n^2-n-4n^2-8n+4}{(n+2)(n+3)^2}$$

$$= \frac{n^3+5n^2+n-8}{(n+2)(n+3)^2}$$

7. Betweenness Centrality of Sunlet Graphs

A sunlet graph is one of the basic graph classes in graph theory. The n-sunlet graph, symbolled by S_n, is formed by attaching the n pendant edges to the cycle C_n. The S_n has $2n$ vertices and $2n$ edges. The S_4 is shown in Figure 6.

Theorem 6 *The $C_B(v)$ of any node point v of the n-sunlet graph S_n is given by*

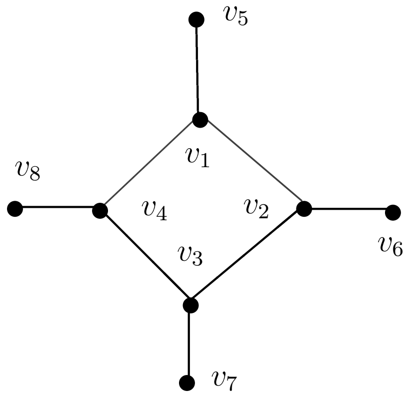

Figure 6: The 4-sunlet graph S_4

.

$$C_B(v) = \begin{cases} 0, & \textit{if } v \textit{ is pendant vertex,} \\ 2(n-1) + \dfrac{(n-2)^2}{2}, & \textit{if } v \in C_n \textit{ and } n \textit{ is even,} \\ 2(n-1) + \dfrac{(n-1)(n-3)}{2}, & \textit{if } v \in C_n \textit{ and } n \textit{ is odd.} \end{cases}$$

.

Proof. Let S_n be the sunlet graph, where $\{v_1, v_2, \cdots, v_n\}$ be the set of vertices of the cycle C_n and $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ are the pendant node points such that the parent node of v_{n+i} is v_i, $i = 1, 2, \ldots, n$. Now, if $v = v_p$, be any vertices of the even cycle C_n then all pairs of vertices (except v_p) of that cycle C_n contribute the value $\dfrac{(n-2)^2}{8}$ [34] to $C_B(v_p)$, and for the odd cycle C_n these pairs contribute the value $\dfrac{(n-1)(n-3)}{8}$ [34] to $C_B(v_p)$. Again, corresponding to each pair of vertices of C_n, there exist another three distinct pairs of vertices (except v_p and v_{n+p}) of S_n which also contribute the same value to $C_B(v)$. For illustration, consider $p \neq 1, n$ and take a pair of vertices of C_n, say (v_{p-1}, v_{p+1}), which contributes the value 1 to $C_B(v)$. Now the corresponding other three pairs of vertices of S_n are (v_{n+p-1}, v_{p+1}), (v_{n+p-1}, v_{n+p+1}) and (v_{p-1}, v_{n+p+1}). Each of these three pairs also contributes the value 1 to $C_B(v)$. So, all pairs of vertices (except v_p and v_{n+p}) of S_n contribute the value $4\frac{(n-2)^2}{8} = \frac{(n-2)^2}{2}$ to $C_B(v_p)$ for the even cycle C_n, and for odd cycle C_n these pairs contribute the value $4\frac{(n-1)(n-3)}{8} = \frac{(n-1)(n-3)}{2}$ to $C_B(v_p)$.

Again, the only shortest path between one of v_{n+p} and one of the other vertices must pass through $v = v_p$. There are $2(n-1)$ such pairs of node points in S_n. Therefore, these pairs of vertices contribute the value $2n-2$ to $C_B(v)$.

If $v = v_{n+i}, i = 1, 2, \cdots, n$ then the shortest routes between any pair of other node points don't pass through v. So, $C_B(v) = 0$. Hence the result is proved. □

Note 13: The relative betweenness centrality of any vertex v is $C'_B(v) = \dfrac{C_B(v)}{MaxC_B(v)}$.

Here, $MaxC_B(v) = \dfrac{(2n-1)(2n-2)}{2}$ (Using the result for the star graph with $2n$ vertices).

So, $C'_B(v) = \dfrac{2C_B(v)}{(2n-1)(2n-2)}$

$$= \begin{cases} 0, & \text{if } v \text{ is pendant vertex,} \\ \dfrac{2}{(2n-1)} + \dfrac{(n-2)^2}{(2n-2)(2n-1)}, & \text{if } v \in C_n \text{ and } n \text{ is even,} \\ \dfrac{2}{(2n-1)} + \dfrac{(n-3)}{2(2n-1)}, & \text{if } v \in C_n \text{ and } n \text{ is odd.} \end{cases}$$

Note 14: For n-sunlet graph, $C_B(v^*) = 2(n-1) + \dfrac{(n-2)^2}{2}$, if n is even

$= 2(n-1) + \dfrac{(n-1)(n-3)}{2}$, if n is odd.

Case 1: When n is even.

$$C_B(S_n) = \frac{2\sum_{i=1}^{2n}\Big[C_B(v^*) - C_B(v_i)\Big]}{(2n-1)^2(2n-2)}$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[\sum_{i=1}^{n}\{C_B(v^*) - C_B(v_i)\} + \sum_{i=n+1}^{2n}\{C_B(v^*) - C_B(v_i)\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[0 + n\{2(n-1) + \frac{(n-2)^2}{2} - 0\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[n\{2(n-1) + \frac{(n-2)^2}{2}\}\Big]$$
$$= \frac{2n\{2(n-1) + \frac{(n-2)^2}{2}\}}{(2n-1)^2(2n-2)}$$

Case 2: When n is odd.

$$C_B(S_n) = \frac{2\sum_{i=1}^{2n}\Big[C_B(v^*) - C_B(v_i)\Big]}{(2n-1)^2(2n-2)}$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[\sum_{i=1}^{n}\{C_B(v^*) - C_B(v_i)\} + \sum_{i=n+1}^{2n}\{C_B(v^*) - C_B(v_i)\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[0 + n\{2(n-1) + \frac{(n-1)(n-3)}{2} - 0\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[n\{2(n-1) + \frac{(n-1)(n-3)}{2}\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[n(n-1)\{2 + \frac{(n-3)}{2}\}\Big]$$
$$= \frac{2}{(2n-1)^2(2n-2)}\Big[n(n-1)\{\frac{4+(n-3)}{2}\}\Big]$$
$$= \frac{2}{2(2n-1)^2(n-1)}\Big[n(n-1)\frac{(n+1)}{2}\Big]$$
$$= \frac{n(n+1)}{2(2n-1)^2}$$

8. Betweenness Centrality of Helm Graphs

The helm graph H_n is formed by joining a pendant node point/vertex to each node point of the largest cycle of a wheel graph W_n with $n+1$ node points, $n \geq 3$ by a bridge. There are $2n+1$ node points and $3n$ edges in H_n. A helm graph H_5 is shown in Figure 7.

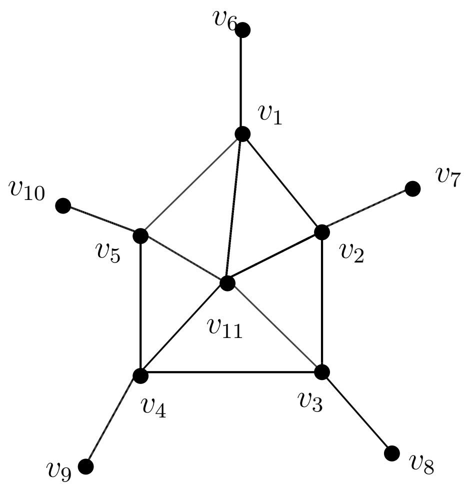

Figure 7: Helm graph H_5

.

Theorem 7 *The $C_B(v)$ of any node point v of helm graph H_n is given by*

$$C_B(v) = \begin{cases} 0, & \text{if } v \text{ is a pendant vertex,} \\ 2n(n-4), & \text{if } v \text{ is the central vertex of } H_n, \\ 2n+1, & \text{otherwise.} \end{cases}$$

.

Proof. Let $v_1, v_2, \cdots, v_n, v_{n+1}, v_{n+2}, \cdots, v_{2n}, v_{2n+1}$ be the node points of H_n, where v_{2n+1} is the central node point of H_n, and $v_1, v_2, \cdots, v_n$ are the node points of the cycle C_n and $v_{n+1}, v_{n+2}, \cdots, v_{2n}$ are the pendant node points of H_n. Suppose v is any vertex/node point of H_n. It is obvious that the betweenness centrality of each pendant node point of the helm graph H_n is zero. If v is any vertex of C_n, say $v = v_p$. Without any loss of generality, we take $p \neq 1, n$. Then there exist two shortest paths between v_{p-1} and v_{p+1}, and one of them passes through v. Also, shortest paths between the other pairs of vertices (except v_p) of W_n do not pass through v. Therefore, the pair of vertices (v_{p-1}, v_{p+1}) contributes the value $\frac{1}{2}$ to $C_B(v)$. Furthermore, each of the pairs of vertices (v_{p-1}, v_{n+p+1}), (v_{n+p-1}, v_{p+1}) and (v_{n+p-1}, v_{n+p+1}) contributes similarly the same value $\frac{1}{2}$ to $C_B(v)$. Again, there exists only one shortest path between v_{n+p} and one of the other vertices of H_n (except v_p), and this path passes through v. The total number of such pairs of vertices is $2n-1$. Therefore $C_B(v) = (2n-1) + 4.\frac{1}{2} = 2n+1$.

Again, for finding $C_B(v_{2n+1})$, all pairs of vertices of the cycle C_n of W_n contribute the value $\frac{n(n-4)}{2}$ [34] to $C_B(v_{2n+1})$. Again, corresponding to each pair of vertices of C_n, there exist three other distinct pairs of vertices of H_n which also contribute the same value to $C_B(v_{2n+1})$. For illustration, take a pair of vertices of C_n, say (v_2, v_n). There are two shortest paths between this pair of vertices, one of which passes through v_{2n+1}. So, this pair contributes the value $\frac{1}{2}$ to $C_B(v_{2n+1})$. Now, corresponding to the pair (v_2, v_n), there exist another three distinct pairs of vertices of H_n, which are (v_2, v_{2n}), (v_{n+2}, v_n) and (v_{n+2}, v_{2n}). Each of these three pairs also contributes the value $\frac{1}{2}$ to $C_B(v_{2n+1})$. So, all pairs of vertices (except v_{2n+1}) of H_n contribute the value $4\frac{n(n-4)}{2}$ to $C_B(v_{2n+1})$, i.e., $C_B(v_{2n+1}) = 2n(n-4)$. So, the result is proved. □

Note 15: The relative betweenness centrality is $C'_B(v) = \frac{C_B(v)}{MaxC_B(v)}$.

In this case, $MaxC_B(v) = \frac{(2n+1-1)(2n+1-2)}{2}$ (Using the result for the star graph with $2n+1$ vertices).

Therefore, $C'_B(v) = \frac{2C_B(v)}{2n(2n-1)} = \frac{C_B(v)}{n(2n-1)}$

$$= \begin{cases} 0, & \text{if } v \text{ is pendant vertex,} \\ \frac{2(n-4)}{2n-1}, & \text{if } v \text{ is central vertex of } H_n, \\ \frac{2n+1}{n(2n-1)}, & \text{otherwise.} \end{cases}$$

Note 16: For H_n, $C_B(v^*) = 2n(n-4)$, $n > 5$.

$$\text{So, } C_B(H_n) = \frac{2\sum_{i=1}^{2n+1}\Big[C_B(v^*) - C_B(v_i)\Big]}{(2n+1-1)^2(2n+1-2)}$$

$$= \frac{2}{4n^2(2n-1)}\Big[\sum_{i=1}^{n}\{2n(n-4)-(2n+1)\} + \sum_{i=n+1}^{2n}\{2n(n-4)-0\}+0\Big]$$

$$= \frac{1}{2n^2(2n-1)}\Big[n\{2n(n-4)\} + \{2n(n-4)-(2n+1)\}\Big]$$

$$= \frac{1}{2n(2n-1)}\Big[2n^2 - 8n + 2n^2 - 8n - 2n - 1\Big]$$

$$= \frac{1}{2n(2n-1)}[4n^2 - 18n - 1]$$

9. Real Applications

Betweenness centrality in friendship and fan graphs identifies key influencers who bridge diverse social clusters to optimize viral marketing. In structured topologies such as lollipop, umbrella, and sunlet graphs, it detects critical ”choke points” or ”stalks” where data flow is most vulnerable to interruption. Similarly, in n-net and helm graphs, it isolates central hubs that manage high-traffic routing, ensuring efficient communication across the network’s periphery. We present a federated learning system using a friendship graph F_n to identify the key node based on betweenness centrality.

9.1 Step-by-Step Representation to Identify the Key Node in Federated Learning System

Step 1: Modeling with Friendship Graphs
We represent a federated learning system using a friendship graph F_n. This graph is constructed by connecting several triangles, where each triangle represents a group of client devices, at a single common vertex. The common vertex corresponds to the central aggregation server, while each triangle represents a local client cluster.
Step 2: Meaning of Nodes and Links

- Vertices inside a triangle represent client devices.
- Edges represent fast and reliable communication links between clients.
- The central vertex represents the global server that collects local updates and redistributes the improved global model.
- There are no direct links between different triangles; hence, clusters communicate only through the central server.

Step 3: Information Flow
If a client in one triangle needs to communicate with a client in another triangle, the communication path must pass through the central server. This accurately reflects the federated learning paradigm, where all model updates and global coordination are mediated by the aggregation server.
Step 4: Betweenness Centrality
Betweenness centrality measures how frequently a node lies on the shortest paths between other pairs of nodes. We can easily compute the betweenness centrality of all the vertices of the modeled friendship graph by our proposed result (1). This measure quantifies the extent to which a node controls the flow of information in the network. 9i9 **Step 5: Centrality in Friendship Graphs**
In the friendship graph F_n, the central server node lies on every shortest path that connects the nodes of different clusters. Consequently, it attains the highest betweenness centrality value, whereas the client nodes within individual triangles exhibit significantly lower centrality.
Step 6: Identification of the Critical Node
The node with the maximum betweenness centrality is identified as the intelligence-critical node. In the context of federated learning, this node corresponds to the aggregation server, as it coordinates all model updates and distributes the global model to client clusters.
Step 7: System-Level Insights
The dominance of the central server in terms of betweenness centrality reveals that:

- It acts as a bottleneck for both communication and computation.
- It plays a crucial role in ensuring the convergence of the learning process.
- It is highly vulnerable to failures or adversarial attacks.

Thus, betweenness centrality serves as a useful metric for evaluating robustness, efficiency, and security in federated learning systems.

Scalability Analysis
To evaluate the scalability of the proposed model, the behavior of betweenness centrality can be examined as the number of client clusters increases. In large-scale friendship graphs, the central aggregation server continues to dominate the shortest communication paths between clusters, leading

to rapid growth in its betweenness centrality. Simulation or analytical results for increasing graph sizes can demonstrate that, while the theoretical characterization remains valid, the communication and computational load on the central server grows significantly with scale. This analysis highlights potential bottlenecks and motivates the need for distributed or multi-server federated learning architectures.

Limitations of the Friendship Graph Model

Despite its analytical usefulness, the friendship graph model has several limitations:

- **Simplified topology:** Real federated learning systems often employ multi-layered or partially decentralized architectures.
- **Single server assumption:** Modern systems may utilize multiple aggregation servers or peer-to-peer coordination.
- **Static structure:** Practical networks are dynamic, with clients joining and leaving over time.
- **Equal edge weights:** Actual communication links differ in bandwidth, latency, and reliability.
- **Scalability issues:** Large-scale federated systems are more complex than simple triangular clusters.
- **Lack of security modeling:** Encryption, secure aggregation, and adversarial behaviors are not explicitly incorporated.

10. Conclusions

In this paper, we investigate the betweenness centrality, relative betweenness centrality, and graph betweenness centrality of several well-known graph classes, namely the friendship graph, fan graph, lollipop graph, umbrella graph, n-net graph, sunlet graph, and helm graph. Exact expressions and theoretical results for these graphs were derived, which help better understand how a graph's structural properties influence the distribution of shortest paths and the relative importance of vertices within a network.

The results obtained provide useful insights into identifying the influential vertices that control information flow in network structures. Such theoretical characterizations are valuable in many practical scenarios where networks must be analyzed or optimized. In particular, the application presented in a federated learning framework demonstrates how betweenness-based measures can help identify critical nodes responsible for communication and aggregation processes in distributed systems.

In general, the study contributes to the theoretical understanding of centrality measures in structured graphs and highlights their relevance in modern network-based applications. As a future direction, the proposed analysis can be extended to more complex settings, such as dynamic graphs and weighted networks, enabling a deeper investigation of evolving real-world systems, including communication networks, social networks, and federated learning environments.

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Conflicts of Interest

The authors declare no conflicts of interest.

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