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An Algebraic Framework for Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets

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ABSTRACT

The extension of algebraic and analytical concepts to the theory of fuzzy sets appears to play a central role in the investigation of nondeterministic techniques. Since an exact description of any real physical situation is virtually impossible, it is necessary to develop schemes which deal analytically with decision processes in an imprecise environment. In this paper, we introduce the concept of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert set and its operations and study some of their structural properties. We also discuss Algebras on Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets, fundamental laws, Algebras on relative complement and Algebra of inclusion with the new operations in a systematic way.

1. Introduction

Modeling uncertainty has long been one of the fundamental challenges in mathematics, computer science, engineering, economics, and decision sciences. Classical mathematical frameworks such as probability theory, interval mathematics, and fuzzy set theory have been extensively employed to represent uncertain and imprecise information. Nevertheless, these approaches often encounter limitations when uncertainty depends on parameterized information or when the available knowledge cannot be adequately characterized by a single mathematical structure. To overcome these shortcomings, Molodtsov introduced the theory of soft sets as a novel parameterized mathematical framework for handling uncertainties that cannot be effectively addressed by conventional mathematical methods [1]. Since its introduction, soft set theory has attracted considerable attention due to its simplicity, flexibility, and broad applicability in decision making, data analysis, engineering, economics, medicine, and numerous other scientific disciplines.

The pioneering work of Maji et al. significantly enriched the theoretical foundation of soft set theory by introducing fundamental concepts such as soft subsets, supersets, null soft sets, absolute

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soft sets, and soft equality, together with several basic operations and the verification of De Morgan's laws [2]. Subsequently, Ali et al. identified inconsistencies in the existing operational definitions and proposed new operations that considerably improved the algebraic consistency of soft set theory [3]. These developments were further extended by investigating various algebraic structures associated with soft sets, thereby establishing a stronger mathematical foundation for parameterized uncertainty modeling [4]. Meanwhile, several researchers explored practical applications of soft sets in decision-making problems and demonstrated their effectiveness in solving complex optimization and selection problems under uncertain environments [5,6]. These contributions established soft set theory as one of the most influential mathematical tools for uncertainty modeling.

To enhance the expressive capability of soft sets, researchers integrated fuzzy set theory with soft sets, resulting in the development of fuzzy soft sets, which effectively combine parameterization with fuzzy membership information [7]. Later, soft sets and fuzzy soft sets were extended to incomplete information systems, providing more realistic representations for practical applications involving missing or uncertain data [8]. More recently, several generalized fuzzy environments have been proposed to improve the representation of uncertainty. Among these, spherical fuzzy sets provide a highly flexible framework by simultaneously considering membership, non-membership, and hesitancy degrees while satisfying relaxed mathematical constraints, thereby extending intuitionistic and picture fuzzy set models [9]. These developments demonstrate the continuous evolution of fuzzy frameworks toward richer and more expressive mathematical models.

Another significant direction in the development of soft set theory concerns its various generalizations. Alkhazaleh et al. introduced the concept of soft multisets to accommodate situations involving multiple occurrences of parameterized information [10]. They further proposed fuzzy parameterized interval-valued fuzzy soft sets and possibility fuzzy soft sets, thereby extending the flexibility of soft set models in uncertainty representation and decision analysis [11,12]. Subsequently, Alkhazaleh and Salleh introduced the notion of soft expert sets, which incorporate the opinions of multiple experts into a unified mathematical framework, allowing decision makers to analyze expert judgments without losing individual assessments [13]. Salleh later presented a comprehensive survey illustrating the development from classical soft sets to intuitionistic fuzzy soft sets and highlighting the increasing importance of parameterized fuzzy models in uncertainty management [14].

The integration of interval-valued fuzzy sets with soft sets further enhanced the capability of uncertainty modeling. Yang et al. introduced interval-valued fuzzy soft sets by combining interval-valued fuzzy information with parameterized soft sets, providing a more flexible representation of uncertain knowledge [15]. The theoretical basis of interval-valued fuzzy sets originated from approximate reasoning under interval uncertainty and has since become an important mathematical tool for modeling imprecise information [17]. Building upon these developments, Jiang et al. proposed interval-valued intuitionistic fuzzy soft sets, which simultaneously incorporate interval-valued membership and non-membership degrees within the soft set framework, thereby offering a more comprehensive representation of uncertainty than conventional fuzzy soft models [18]. Owing to their enhanced expressive capability, interval-valued intuitionistic fuzzy soft sets have been successfully employed in multiple attribute decision-making problems, pattern recognition, and information processing.

To further improve the flexibility of parameterized fuzzy models, Majumdar and Samanta introduced generalized fuzzy soft sets by associating a degree with each parameter, thereby enriching the parameterization process and increasing the descriptive power of fuzzy soft sets [19]. Inspired by this idea, Babitha and John extended the concept to generalized intuitionistic fuzzy soft sets and demonstrated its applicability to multicriteria decision-making problems [20]. Furthermore, Alkhazaleh and Salleh proposed fuzzy soft expert sets, integrating expert opinions with fuzzy soft information to provide more comprehensive support for intelligent decision-making systems [21], while Qayyum et al. later introduced cubic soft expert sets to further generalize expert-based fuzzy environments [22]. Another important extension was presented through generalized interval-valued fuzzy soft sets, which combine generalized parameterization with interval-valued fuzzy information and provide an effective mathematical framework for handling more complex uncertain information [23].

The rapid advancement of generalized fuzzy theories has significantly expanded the mathematical foundations available for modeling uncertainty, vagueness, and imprecision in intelligent information systems. In recent years, considerable attention has been devoted to developing generalized fuzzy environments capable of handling increasingly complex forms of uncertain information while simultaneously improving the reliability of decision-support methodologies. Among these developments, similarity measures for generalized interval-valued intuitionistic fuzzy soft expert sets have demonstrated remarkable effectiveness in medical diagnosis by providing accurate comparisons between uncertain expert opinions and clinical information [24]. This work highlights the practical importance of generalized interval-valued intuitionistic fuzzy soft expert models and illustrates their potential for solving real-world problems involving uncertain expert knowledge.

Beyond similarity analysis, the development of advanced aggregation operators has become another important research direction in generalized fuzzy decision-making. Aggregation techniques provide effective mechanisms for combining uncertain information originating from multiple sources, attributes, or experts while preserving the underlying uncertainty. Recent studies have proposed several aggregation operators under generalized hesitant fuzzy, intuitionistic hesitant fuzzy, and q-rung orthopair fuzzy environments, substantially improving the effectiveness and robustness of multiple-attribute decision-making methodologies [30,35]. Likewise, generalized hesitant fuzzy frameworks have been successfully applied to sustainable supply chain optimization [27], while novel intuitionistic hesitant fuzzy group decision-making models and Fermatean fuzzy Aczel–Alsina aggregation operators have provided efficient mathematical tools for investment policy selection and strategic investment analysis under uncertain environments [34,36]. These studies clearly demonstrate that generalized fuzzy structures continue to play an increasingly significant role in modern intelligent decision-support systems.

Parallel to these developments, substantial progress has also been achieved in generalized rough set theory, fuzzy graph theory, and algebraic fuzzy systems. Recent sequential decision-theoretic rough set models have provided more flexible approximation mechanisms for uncertain decision environments by extending the classical rough set framework [31]. Similarly, dominance-based fuzzy rough set models have enhanced classification and knowledge discovery through improved approximation operators [16], whereas multi-granulation fuzzy rough set environments have further strengthened uncertainty analysis by considering multiple information granules simultaneously [37]. In addition, fuzzy graph theory has experienced considerable advancement through the introduction of directed fuzzy edge graph models capable of addressing optimal pathfinding problems under q-rung orthopair fuzzy environments [28]. From an algebraic perspective, recent investigations of Fermatean fuzzy algebraic structures have also enriched the mathematical theory of generalized fuzzy systems by establishing new algebraic characterizations within fuzzy ring structures [32]. Collectively, these contributions illustrate the continuous integration of generalized fuzzy algebra with intelligent computation, optimization, and uncertainty modeling.

Recent advances further demonstrate that generalized fuzzy methodologies have become indispensable tools for solving challenging problems across artificial intelligence, machine learning, and intelligent information processing. Generalized fuzzy techniques have recently been employed in DeepFake detection to improve the reliability of multimedia authentication [25], while deep learning models integrated with intelligent uncertainty-handling mechanisms have significantly improved foreign object debris detection for aviation safety applications [26]. Likewise, generalized fuzzy reasoning has contributed to the development of cooperative heterogeneous multi-agent reinforcement learning systems capable of making reliable decisions in dynamic uncertain environments [29]. These emerging applications further confirm that the success of intelligent systems increasingly depends on mathematically rigorous uncertainty models capable of integrating multiple sources of imprecise information.

The versatility of generalized fuzzy methodologies has also been demonstrated in contemporary sustainability and intelligent decision-making applications. Recent fuzzy multi-criteria decision-making frameworks have been successfully employed for assessing carbon border adjustment mechanism readiness [38], prioritizing resilient vaccine distribution strategies [39], evaluating air quality indices using T-spherical fuzzy environments [40], developing emotionally intelligent artificial intelligence through complex circular intuitionistic fuzzy frameworks [41], constructing exponential similarity measures under q-rung orthopair spherical fuzzy environments [42], assessing tax performance using grey hybrid decision-making models [43], identifying traffic accident hotspots through fuzzy GIS-based methodologies [44], evaluating socio-economic sustainability in urban transportation systems [45], and assessing barriers to sustainable energy transition using fuzzy multi-criteria decision-making techniques [46]. These studies collectively illustrate the remarkable diversity of generalized fuzzy methodologies and their growing importance in addressing complex decision-making problems across numerous scientific disciplines.

Despite these remarkable developments, an important theoretical limitation remains. Existing studies have primarily focused on designing similarity measures, aggregation operators, decision-making algorithms, optimization techniques, and application-oriented models for generalized fuzzy environments. Comparatively little attention has been devoted to investigating the fundamental algebraic structures underlying generalized interval-valued intuitionistic fuzzy soft expert sets. In particular, the literature lacks a comprehensive mathematical framework that systematically defines the basic operations, establishes their algebraic properties, investigates inclusion relationships, and develops the associated algebraic laws governing generalized interval-valued intuitionistic fuzzy soft expert sets. Consequently, the absence of a rigorous algebraic foundation restricts both the mathematical understanding of these models and their future theoretical extensions.

This limitation provides the principal motivation for the present study. Rather than proposing another application-specific decision-making model or introducing a new aggregation operator, this work focuses on establishing the underlying algebraic theory of generalized interval-valued intuitionistic fuzzy soft expert sets. By constructing a rigorous algebraic framework, the proposed model aims to provide the mathematical infrastructure required for future developments in generalized fuzzy theory and expert-oriented uncertainty modeling. Such a framework is expected to facilitate the systematic development of new similarity measures, aggregation operators, ranking methodologies, optimization techniques, and intelligent decision-support models while ensuring their mathematical consistency. The main contributions of this paper are summarized as follows:

i. We introduce the concept of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GIVIFSESs) by extending generalized interval-valued fuzzy soft sets to the interval-valued intuitionistic fuzzy soft expert environment.

ii. We develop a comprehensive collection of fundamental operations for GIVIFSESs, including sum, product, subtraction, max-product, min-product, complement, power, and scalar product.
iii. We establish a rigorous algebraic framework by investigating fundamental algebraic laws, inclusion relations, relative complements, and other structural properties of GIVIFSESs.
iv. We provide a unified mathematical foundation that not only generalizes several existing fuzzy soft expert models but also facilitates future theoretical developments and advanced intelligent decision-support methodologies.

The remainder of this paper is organized as follows. Section 2 reviews the preliminary concepts and mathematical definitions required throughout the paper. Section 3 introduces Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets together with their fundamental operations and structural properties. Section 4 develops the proposed algebraic framework by establishing the principal algebraic laws, inclusion relations, relative complements, and other essential theoretical results. Finally, Section 5 concludes the paper and discusses several promising directions for future research.

2. Preliminaries

In this section, we recall some basic definitions such as soft set, soft expert set, fuzzy soft expert set, interval-valued fuzzy set, interval-valued intuitionistic fuzzy soft expert set and generalized interval-valued fuzzy soft set which are required in the construction of new concept.

Definition 2.1 [1] *Let* $\widehat{F}: \mathcal{A} \to P(U)$ is a mapping, whose domain is a set of parameters and codomain is a power set of universal setU. **Soft set** over the universal set U is represented as a pair $\widehat{(\mathrm{F}: \mathcal{A})}$. For $\propto \in \mathcal{A}$, $\widehat{\mathrm{F}}(\propto)$ gives the set of $\propto$-approxinmate elements of the soft set $\widehat{(\mathrm{F}: \mathcal{A})}$. It is considered as parameterized family of subsets of the set U.

Definition 2.2 [13] Let $\widehat{F}: \mathcal{A} \to P(U)$ is a mapping, whose domain is cartesian product of set of parameters, set of experts and set of opinions i-e $\mathrm{Ą} = \hat{\mathrm{E}} \times \mathrm{Ķ} \times \mathrm{Ợ}$ and codomain is a power set of universal set U. **Soft expert set** over the universal set U is represented as a pair $\widehat{(\mathrm{F}: \mathcal{A})}$.

Definition 2.3 [21] Let $\widehat{F}: \mathcal{A} \to I^U$ is a mapping, whose domain is cartesian product of set of parameters, set of experts and set of opinions i-e $\mathrm{Ą} = \hat{\mathrm{E}} \times \mathrm{Ķ} \times \mathrm{Ợ}$ and codomain I^Udenoted all fuzzy subsets of U is a power sets of universal set U. **Fuzzy soft expert set** over the universal set U is represented as a pair $\widehat{(\mathrm{F}: \mathcal{A})}$.

Definition 2.4 [15] Let U be a non-empty set. A function $\mathrm{Ą}: \mathrm{U} \to \mathrm{Int}[0,1]$ is called an **interval-valued fuzzy set**, whose domain is a universal set and codomain is the set of all closed sub intervals of [0,1], the set of all interval-valued fuzzy sets on U is denoted by I^U. For every $\mathrm{Ą} \in I^U$and u$\in U$, $\mathrm{Ą}(\mathrm{u}) = [\mathrm{Ą}^-(u), \mathrm{Ą}^+(u)]$ is called the degree of membership of an element u to Ą, lower fuzzy set and upper fuzzy set in U represented by the following maps $\mathrm{Ą}^-: U \to I$ and $\mathrm{Ą}^+: U \to I$ respectively.

Definition 2.5 [22] Let $\xi: \mathrm{Ą} \times \mathrm{Ģ} \to \mathrm{Ķ}(\mathrm{U})$ is a mapping, whose domain is a cartesian product of the set of attributes Ą and the set of experts Ģ and codomain in the set of the interval-valued intuitionistic fuzzy sets on the universe set U which is denoted as Ķ(U). **Interval -valued intuitionistic fuzzy soft expert set** (IVIFSE set) is a triplet$(\xi: \mathrm{Ą}, \mathrm{Ģ})$. For b∈ Ą and p∈ Ģ we define

$$\xi_p^b = \left\{< u, \left[{\gamma^-}_p^b(u), {\gamma^+}_p^b(u)\right], \left[{\zeta^-}_p^b(u), {\zeta^+}_p^b(u)\right] >: u \in U\right\}.$$

Definition 2.6 [23] Let $\check{\mathrm{U}} = \{u_1, u_2, u_3 \ldots\ldots u_n\}$ be the universal set of elements and $\hat{\mathrm{E}} = \{v_1, v_2, v_3 \ldots\ldots v_n\}$ be the universal set of parameters. The pair $(\check{\mathrm{U}}, \hat{\mathrm{E}})$ will be called a soft universe.

Let $\digamma: \hat{E} \rightarrow Int(\breve{U})$ and η be a fuzzy set of $\hat{E}$ i.e $\eta: \hat{E} \rightarrow I = [0,1]$, where $Int(\breve{U})$ is the set of all interval-valued fuzzy subsets on $\breve{U}$. Let $\digamma_\eta: \hat{E} \rightarrow Int(\breve{U}) \times I$ be a function defined as follows:

$$\digamma_\eta(v) = \left(\digamma(v)(u), \eta(v)\right), \forall u \in \breve{U}.$$

Then $\digamma_\eta$ is called a **generalized interval-valued fuzzy soft set** (GIVFS set) over the soft set $(\breve{U}, \hat{E})$.

Definition 2.7 [24] If m (f, g) be the similarity measure between two interval valued intuitionistic fuzzy soft sets (F, A) and (G, B) then,

1. $m(f, g) = m(g, f)$
2. $0 \leq m(f, g) \leq 1$
3. $m(f, g) = 1 \Leftrightarrow (F, A) = (G, B)$

3. The Proposed concept of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets and Their Fundamental Operations

The terms Generalized Interval-Valued Fuzzy Soft set and Interval-Valued Intuitionistic Fuzzy Soft Expert set are combined in this section to create the new concept known as Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets. We define the idea of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets and define some related concepts pertaining to this notion as well as their basic operations on this concept.

Definition 3.1 Let $\hat{U} = \{\xi_1, \xi_2, \xi_3, \ldots\ldots \xi_n\}$ be the initial universal set of elements, $\hat{A} = \{\sigma_1, \sigma_2, \sigma_3 \ldots\ldots \sigma_n\}$ be the set of attributes and $\hat{G} = \{\rho_1, \rho_2, \rho_3 \ldots\ldots \rho_n\}$ be the set of experts. The Interval-Valued Intuitionistic Fuzzy Soft Expert set is a triplet $(\hat{U}, \hat{A}, \hat{G})$ that is described by a mapping $£: \hat{A} \times \hat{G} \rightarrow Ķ(\hat{U})$ where $Ķ(\hat{U})$ is the set of all Interval-Valued Intuitionistic Fuzzy subset on $\hat{U}$. Let $£: \hat{A} \times \hat{G} \rightarrow Ķ(\hat{U})$ and η be a fuzzy soft expert set of $\hat{A} \times \hat{G}$, i.e., $\eta: \hat{A} \times \hat{G} \rightarrow I = [0,1]$. Let $£_\eta: \hat{A} \times \hat{G} \rightarrow Ķ(\hat{U}) \times I$ be a function defined as follow:

$$£_{\eta_\sigma}^{\rho} = \left(\{< \xi, [\alpha^{-\rho}_{\sigma}, \alpha^{+\rho}_{\sigma}(u)], [\beta^{-\rho}_{\sigma}, \beta^{+\rho}_{\sigma}] >\}, \eta^{\rho}_{\sigma}(\xi)\right),$$

Then $£_{\eta_\sigma}^{\rho}$ is called a Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert set.

GIVIFSEs over the soft expert set $(\hat{U}, \hat{A}, \hat{G})$.

Example 3.1 Let $\hat{U} = \{\xi_1, \xi_2, \xi_3\}$ be the set of universes , $\hat{A} = \{\sigma_1, \sigma_2, \sigma_3\}$ be the set of attributes and $\hat{G} = \{\rho_1, \rho_2\}$ be the set of experts and $\eta: \hat{A} \times \hat{G} \rightarrow I$ define a function $£_\eta: \hat{A} \times \hat{G} \rightarrow Ķ(\hat{U}) \times I$ as follows:

$$£_{\eta_{\sigma 1}}^{\rho 1} = \left(\left\{\frac{\xi_1}{[0.2,0,4], [0.1,0.3]}, \frac{\xi_2}{[0.1,0.5], [0.4,0.5]}, \frac{\xi_3}{[0.3,0.4], [0.4,0.5]}\right\}, 0.5\right),$$

$$£_{\eta_{\sigma 2}}^{\rho 1} = \left(\left\{\frac{\xi_1}{[0.3,0,4], [0.1,0.2]}, \frac{\xi_2}{[0.5,0.6], [0.4,0.6]}, \frac{\xi_3}{[0.2,0.4], [0.3,0.5]}\right\}, 0.6\right),$$

$$£_{\eta_{\sigma 3}}^{\rho 1} = \left(\left\{\frac{\xi_1}{[0.2,0,5], [0.1,0.3]}, \frac{\xi_2}{[0.4,0.6], [0.5,0.7]}, \frac{\xi_3}{[0.1,0.2], [0.3,0.4]}\right\}, 0.3\right),$$

$$£_{\eta_{\sigma 1}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.1,0,3], [0.4,0.6]}, \frac{\xi_2}{[0.5,0.6], [0.1,0.3]}, \frac{\xi_3}{[0.7,0.8], [0.1,0.2]}\right\}, 0.4\right),$$

$$£_{\eta_{\sigma 2}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.0,0.4], [0.0,0.3]}, \frac{\xi_2}{[0.1,0.2], [0.6,0.7]}, \frac{\xi_3}{[0.1,0.5], [0.0,0.3]}\right\}, 0.1\right),$$

$$£_{\eta_{\sigma 3}}^{\rho 2} = \left(\left\{\frac{\xi_1}{[0.1,0.4], [0.1,0.2]}, \frac{\xi_2}{[0.2,0.5], [0.0,0.5]}, \frac{\xi_3}{[0.3,0.5], [0.0,0.1]}\right\}, 0.7\right),$$

Then $£_\eta$ is called a GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. In matrix notation we write in Table 1 and Table 2.

Table 1
Opinion of expert ρ_1

ρ_1	ξ_1	ξ_2	ξ_3	$\eta^{\rho}_{\sigma}(\xi)$
σ_1	[0.2,0,4], [0.1,0.3]	[0.1,0.5], [0.4,0.5]	[0.3,0.4], [0.4,0.5]	0.5
σ_2	[0.3,0,4], [0.1,0.2]	[0.5,0.6], [0.4,0.6]	[0.2,0.4], [0.3,0.5]	0.6
σ_3	[0.2,0,5], [0.1,0.3]	[0.4,0.6], [0.5,0.7]	[0.1,0.2], [0.3,0.4]	0.3

Table 2
Opinion of expert ρ_2

ρ_2	ξ_1	ξ_2	ξ_3	$\eta^{\rho}_{\sigma}(\xi)$
σ_1	[0.1,0,3], [0.4,0.6]	[0.5,0.6], [0.1,0.3]	[0.7,0.8], [0.1,0.2]	0.5
σ_2	[0.0,0.4], [0.0,0.3]	[0.1,0.2], [0.6,0.7]	[0.1,0.5], [0.0,0.3]	0.6
σ_3	[0.1,0.4], [0.1,0.2]	[0.2,0.5], [0.0,0.5]	[0.3,0.5], [0.0,0.1]	0.3

Definition 3.2 Let $£_{\eta_1}$ and Γ_{η_2} be two GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. Then $£_{\eta_1}$ is called a *GIVIFSE* subset of Γ_{η_2} and we write $£_{\eta_1} \subseteq \Gamma_{\eta_2}$ if.

a) $\eta_1{}^{\rho}_{\sigma}(\xi)$ is a fuzzy subset of $\eta_2{}^{\rho}_{\sigma}(\xi)\ \forall\, \xi \in \hat{U}$,
b) $£_{\eta^{\rho}_{\sigma}}$ is an IVIF subset of $\Gamma_{\eta^{\rho}_{\sigma}}$, $\forall\, \sigma \in \hat{A}\ and\ \rho \in \hat{G}$.

Definition 3.3 The complement of GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$ is denoted by $(\hat{U}, \hat{A}, \hat{G})^{ç}$ and is defined for all $\sigma \in \hat{A}\ and\ \rho \in \hat{G}$ as follows:

$(\hat{U}, \hat{A}, \hat{G})^{ç} = \left(\left\{< \xi, \left[\beta^{-\rho}_{\ \sigma}, \beta^{+\rho}_{\ \sigma}\right], \left[\alpha^{-\rho}_{\ \sigma}, \alpha^{+\rho}_{\ \sigma}(u)\right] >\right\}, 1 - \eta^{\rho}_{\sigma}(\xi)\right)$,

Where $£_{\eta^{\rho}_{\sigma}} = \left(\left\{< \xi, \left[\alpha^{-\rho}_{\ \sigma}, \alpha^{+\rho}_{\ \sigma}(u)\right], \left[\beta^{-\rho}_{\ \sigma}, \beta^{+\rho}_{\ \sigma}\right] >\right\}, \eta^{\rho}_{\sigma}(\xi)\right)$.

Definition 3.4 Let $£_{\eta_1}$ be a GIVIFSEs over $(\hat{U}, \hat{A}, \hat{G})$. Then the complement of $£_{\eta_1}$ denoted by $£_{\eta_1}{}^{ç}$ and is defined by $£_{\eta_1}{}^{ç} = \Gamma_{\eta}$ such that $\eta_2{}^{\rho}_{\sigma}(\xi) = ç\,(\eta_1{}^{\rho}_{\sigma}(\xi))$ and $\Gamma^{\rho}_{\sigma} = ç\,(£^{\rho}_{\sigma})\ \forall\, \sigma \in \hat{A}\ and\ \rho \in \hat{G}$ where $ç$ is a fuzzy complement and ĉ is an Interval-Valued Intuitionistic Fuzzy complement.

Definition 3.5 The sum of two GIVIFSEs $(£_1, \hat{A}_1, \hat{G}_1)$ and $(£_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(£_1, \hat{A}_1, \hat{G}_1) + (£_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$£^{\rho}_{1\sigma} + £^{\rho}_{2\sigma} = \left(\begin{array}{c}\left\{\begin{array}{c}< \xi, \begin{bmatrix}\alpha_1{}^{-\rho}_{\ \sigma}(\xi) + \alpha_2{}^{-\rho}_{\ \sigma}(\xi) - \alpha_1{}^{-\rho}_{\ \sigma}(\xi)\alpha_2{}^{-\rho}_{\ \sigma}(\xi), \\ \alpha_1{}^{+\rho}_{\ \sigma}(\xi) + \alpha_2{}^{+\rho}_{\ \sigma}(\xi) - \alpha_1{}^{+\rho}_{\ \sigma}(\xi)\alpha_2{}^{+\rho}_{\ \sigma}(\xi),\end{bmatrix}, \\ \left[\beta_1{}^{-\rho}_{\ \sigma}(\xi)\beta_2{}^{-\rho}_{\ \sigma}(\xi), \beta_1{}^{+\rho}_{\ \sigma}(\xi)\beta_2{}^{+\rho}_{\ \sigma}(\xi)\right], >\end{array}\right\}, \\ \eta_1{}^{\rho}_{\sigma}(\xi) + \eta_2{}^{\rho}_{\sigma}(\xi) - \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi)\end{array}\right),$$

Where $£_{\eta^{\rho}_{1\sigma}} = \left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\ \sigma}, \alpha_1{}^{+\rho}_{\ \sigma}(u)\right], \left[\beta_1{}^{-\rho}_{\ \sigma}, \beta_1{}^{+\rho}_{\ \sigma}\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)$,

and $£_{\eta^{\rho}_{2\sigma}} = \left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\ \sigma}, \alpha_2{}^{+\rho}_{\ \sigma}(u)\right], \left[\beta_2{}^{-\rho}_{\ \sigma}, \beta_2{}^{+\rho}_{\ \sigma}\right] >\right\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)$.

Definition 3.6 The difference of two GIVIFSEs $(£_1, \hat{A}_1, \hat{G}_1)$ and $(£_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(£_1, \hat{A}_1, \hat{G}_1) - (£_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$£_1{}^{\rho}_{\sigma} - £_2{}^{\rho}_{\sigma} = \left(\begin{array}{c}\left\{\begin{array}{c}< \xi, \left[\alpha_1{}^{-\rho}_{\ \sigma}(\xi) \wedge \beta_2{}^{-\rho}_{\ \sigma}(\xi), \alpha_1{}^{+\rho}_{\ \sigma}(\xi) \wedge \beta_2{}^{+\rho}_{\ \sigma}(\xi)\right], \\ \left[\beta_1{}^{-\rho}_{\ \sigma}(\xi) \vee \alpha_2{}^{-\rho}_{\ \sigma}(\xi), \beta_1{}^{+\rho}_{\ \sigma}(\xi) \vee \alpha_1{}^{+\rho}_{\ \sigma}(\xi)\right] >\end{array}\right\}, \\ \eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\end{array}\right),$$

Where $£_{\eta^{\rho}_{1\sigma}} = \left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\ \sigma}, \alpha_1{}^{+\rho}_{\ \sigma}(u)\right], \left[\beta_1{}^{-\rho}_{\ \sigma}, \beta_1{}^{+\rho}_{\ \sigma}\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)$, and $£_{\eta_{2\sigma}}^{\rho} = \left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)\right], \left[\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}\right] >\right\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)$.

Definition 3.7 The product of two GIVIFSEs $(£_1, \hat{A}_1, \hat{G}_1)$ and $(£_2, \hat{A}_2, \hat{G}_2)$ over ξ is denoted as $(£_1, \hat{A}_1, \hat{G}_1) \times (£_2, \hat{A}_2, \hat{G}_2)$ and defined as:

$$£_{1\sigma}^{\rho} £_{2\sigma}^{\rho} = \left(\begin{array}{c} \left\{ \begin{array}{c} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma}(\xi)\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)\alpha_2{}^{+\rho}_{\sigma}(\xi)\right], \\ \left[\begin{array}{c} \beta_1{}^{-\rho}_{\sigma}(\xi) + \beta_2{}^{-\rho}_{\sigma}(\xi) - \beta_1{}^{-\rho}_{\sigma}(\xi)\beta_2{}^{-\rho}_{\sigma}(\xi), \\ \beta_1{}^{+\rho}_{\sigma}(\xi) + \beta_2{}^{+\rho}_{\sigma}(\xi) - \beta_1{}^{+\rho}_{\sigma}(\xi)\beta_2{}^{+\rho}_{\sigma}(\xi) \end{array} \right] > \end{array} \right\}, \\ \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \end{array} \right),$$

Where $£_{\eta_{1\sigma}}^{\rho} = \left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\right], \left[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)$

and $£_{\eta_{2\sigma}}^{\rho} = \left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)\right], \left[\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}\right] >\right\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)$

Definition 3.8 The scalar product of GIVIFSEs $(\xi, \hat{A}, \hat{G})$ with an arbitrary real number $ɽ > 0$ is denoted by $ɽ(\xi, \hat{A}, \hat{G})$ and defined as follow:

$$ɽ£_{\sigma}^{\rho} = \left(\begin{array}{c} \left\{< \xi, \left[1 - \left(1 - \alpha^{-\rho}_{\sigma}\right)^{ɽ}, 1 - \left(1 - \alpha^{+\rho}_{\sigma}\right)^{ɽ}\right], \left[(\beta^{-\rho}_{\sigma})^{ɽ}, (\beta^{+\rho}_{\sigma})^{ɽ}\right] >, \right\} \\ 1 - (1 - \eta_{\sigma}^{\rho})^{ɽ} \end{array} \right)$$

Definition 3.9 The power of GIVIFSEs$(\xi, \hat{A}, \hat{G})$ with an arbitrary real number $ɽ > 0$ is denoted by $(\xi, \hat{A}, \hat{G})^{ɽ}$ over ξ and defined as follow:

$$(£_{\sigma}^{\rho})^{ɽ} = \left(\begin{array}{c} \left\{< \xi, \left[(\alpha^{-\rho}_{\sigma})^{ɽ}, (\alpha^{+\rho}_{\sigma})^{ɽ}\right]\left[1 - \left(1 - \beta^{-\rho}_{\sigma}\right)^{ɽ}, 1 - \left(1 - \beta^{+\rho}_{\sigma}\right)^{ɽ}\right] >, \right\} \\ (\eta_{\sigma}^{\rho})^{ɽ} \end{array} \right)$$

4. Algebra on Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert sets:

4.1 Introduction

There are numerous algebras connected to logic. Aristotelean logic with two values and standard Boolean algebras are related. Multivalued logic works well with MV algebras. In addition to generalizing implication algebras, BCI/BCK algebras also generalize the idea of the algebra of sets, where the only non-null operation is set subtraction. In this chapter we investigate certain algebras on GIV IF SE sets with respect to different operations.

Proposition 1. Let

$£_{\eta_{1\sigma_1}}^{\rho_1} = \left(\left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_1{}^{-\rho}_{\sigma}(\xi), \beta_1{}^{+\rho}_{\sigma}(\xi)\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_1}_{\sigma_1} \in \hat{A}_1 \times \hat{G}_1\right)$,

$\Gamma_{\eta_{2\sigma_2}}^{\rho_2} = \left(\left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_2{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_2{}^{-\rho}_{\sigma}(\xi), \beta_2{}^{+\rho}_{\sigma}(\xi)\right] >\right\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_2}_{\sigma_2} \in \hat{A}_2 \times \hat{G}_2\right)$,

be two GIVIFSESs over ξ. Then commutative law with respect to product hold:

$$£_{\eta_{1\sigma_1}}^{\rho_1} \otimes \Gamma_{\eta_{2\sigma_2}}^{\rho_2} = \Gamma_{\eta_{2\sigma_2}}^{\rho_2} \otimes £_{\eta_{1\sigma_1}}^{\rho_1}$$

Proof.

$$£_{\eta_{1\sigma_1}}^{\rho_1} \otimes \Gamma_{\eta_{2\sigma_2}}^{\rho_2}$$

$$= \left[\begin{array}{c} \left(\left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_1{}^{-\rho}_{\sigma}(\xi), \beta_1{}^{+\rho}_{\sigma}(\xi)\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_1}_{\sigma_1} \in \hat{A}_1 \times \hat{G}_1\right) \otimes \\ \left(\left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_2{}^{+\rho}_{\sigma}(\xi)\right], \left[\beta_2{}^{-\rho}_{\sigma}(\xi), \beta_2{}^{+\rho}_{\sigma}(\xi)\right] >\right\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right): \xi \in \hat{U}, {}^{\rho_2}_{\sigma_2} \in \hat{A}_2 \times \hat{G}_2\right) \end{array} \right],$$

$$= \left[\begin{array}{c} \left\{ \begin{array}{c} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma}(\xi)\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)\alpha_2{}^{+\rho}_{\sigma}(\xi)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma}(\xi) + \beta_2{}^{-\rho}_{\sigma}(\xi) - \beta_1{}^{-\rho}_{\sigma}(\xi)\beta_2{}^{-\rho}_{\sigma}(\xi), \beta_1{}^{+\rho}_{\sigma}(\xi) + \beta_2{}^{+\rho}_{\sigma}(\xi) - \beta_1{}^{+\rho}_{\sigma}(\xi)\beta_2{}^{+\rho}_{\sigma}(\xi)\right] > \end{array} \right\}, \\ \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \end{array} \right],$$

$$= \left[\begin{Bmatrix} <\xi, \left[{\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi){\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi), {\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi){\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi) \right], \\ \left[{\beta_2}^{-}{}^{\rho}_{\sigma}(\xi) + {\beta_1}^{-}{}^{\rho}_{\sigma}(\xi) - {\beta_2}^{-}{}^{\rho}_{\sigma}(\xi){\beta_1}^{-}{}^{\rho}_{\sigma}(\xi), {\beta_2}^{+}{}^{\rho}_{\sigma}(\xi) + {\beta_1}^{+}{}^{\rho}_{\sigma}(\xi) - {\beta_2}^{+}{}^{\rho}_{\sigma}(\xi){\beta_1}^{+}{}^{\rho}_{\sigma}(\xi) \right] > \end{Bmatrix}, \\ \eta_2{}^{\rho}_{\sigma}(\xi)\eta_1{}^{\rho}_{\sigma}(\xi) \right],$$
$$= \Gamma_{\eta_{2\sigma_2}}^{\rho_2} \otimes £_{\eta_{1\sigma_1}}^{\rho_1}$$

Proposition 2. Let $£_{\eta_{1\sigma}}^{\rho}$ and $£_{\eta_{2\sigma}}^{\rho}$ are two GIVIFSE sets. Then De Morgan's laws with respect to addition and multiplication also hold:

i. $\left(£_{\eta_{1\sigma}}^{\rho} \oplus £_{\eta_{2\sigma}}^{\rho}\right)^{c} = \left(£_{\eta_{1\sigma}}^{\rho}\right)^{c} \otimes \left(£_{\eta_{2\sigma}}^{\rho}\right)^{c}$.

ii. $\left(£_{\eta_{1\sigma}}^{\rho} \otimes £_{\eta_{2\sigma}}^{\rho}\right)^{c} = \left(£_{\eta_{1\sigma}}^{\rho}\right)^{c} \oplus \left(£_{\eta_{2\sigma}}^{\rho}\right)^{c}$.

Proof. i)

$$£_{\eta_{1\sigma}}^{\rho} = \left(\left\{ <\xi, \left[{\alpha_1}^{-}{}^{\rho}_{\sigma}, {\alpha_1}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_1}^{-}{}^{\rho}_{\sigma}, {\beta_1}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right),$$
$$\text{and } £_{\eta_{2\sigma}}^{\rho} = \left(\left\{ <\xi, \left[{\alpha_2}^{-}{}^{\rho}_{\sigma}, {\alpha_2}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_2}^{-}{}^{\rho}_{\sigma}, {\beta_2}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \right),$$
$$\left(£_{\eta_{1\sigma}}^{\rho} \oplus £_{\eta_{2\sigma}}^{\rho}\right)^{c}$$
$$= \begin{pmatrix} \left(\left\{ <\xi, \left[{\alpha_1}^{-}{}^{\rho}_{\sigma}, {\alpha_1}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_1}^{-}{}^{\rho}_{\sigma}, {\beta_1}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right) \oplus \\ \left(\left\{ <\xi, \left[{\alpha_2}^{-}{}^{\rho}_{\sigma}, {\alpha_2}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_2}^{-}{}^{\rho}_{\sigma}, {\beta_2}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \right) \end{pmatrix}^{c},$$
$$= \left(\begin{pmatrix} \begin{Bmatrix} <\xi, \begin{bmatrix} {\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi) + {\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi) - {\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi), \\ {\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi) + {\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi) - {\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi), \end{bmatrix}, \\ \left[{\beta_1}^{-}{}^{\rho}_{\sigma}(\xi){\beta_2}^{-}{}^{\rho}_{\sigma}(\xi), {\beta_1}^{+}{}^{\rho}_{\sigma}(\xi){\beta_2}^{+}{}^{\rho}_{\sigma}(\xi) \right], > \end{Bmatrix}, \\ \eta_1{}^{\rho}_{\sigma}(\xi) + \eta_2{}^{\rho}_{\sigma}(\xi) - \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \end{pmatrix} \right)^{c},$$
$$= \begin{pmatrix} \begin{Bmatrix} <\xi, \left[{\beta_1}^{-}{}^{\rho}_{\sigma}(\xi){\beta_2}^{-}{}^{\rho}_{\sigma}(\xi), {\beta_1}^{+}{}^{\rho}_{\sigma}(\xi){\beta_2}^{+}{}^{\rho}_{\sigma}(\xi) \right], \\ \begin{bmatrix} {\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi) + {\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi) - {\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi), \\ {\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi) + {\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi) - {\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi), \end{bmatrix}, > \end{Bmatrix}, \\ 1 - \eta_1{}^{\rho}_{\sigma}(\xi) + \eta_2{}^{\rho}_{\sigma}(\xi) - \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \end{pmatrix},$$
$$= \begin{pmatrix} \left(\left\{ <\xi, \left[{\beta_1}^{-}{}^{\rho}_{\sigma}, {\beta_1}^{+}{}^{\rho}_{\sigma} \right], \left[{\alpha_1}^{-}{}^{\rho}_{\sigma}, {\alpha_1}^{+}{}^{\rho}_{\sigma}(u) \right] > \right\}, 1 - \eta_1{}^{\rho}_{\sigma}(\xi) \right) \otimes \\ \left(\left\{ <\xi, \left[{\beta_2}^{-}{}^{\rho}_{\sigma}, {\beta_2}^{+}{}^{\rho}_{\sigma} \right], \left[{\alpha_2}^{-}{}^{\rho}_{\sigma}, {\alpha_2}^{+}{}^{\rho}_{\sigma}(u) \right], > \right\}, 1 - \eta_2{}^{\rho}_{\sigma}(\xi) \right) \end{pmatrix},$$
$$= \left(£_{\eta_{1\sigma}}^{\rho}\right)^{c} \otimes \left(£_{\eta_{2\sigma}}^{\rho}\right)^{c}.$$

ii) $\left(£_{\eta_{1\sigma}}^{\rho} \otimes £_{\eta_{2\sigma}}^{\rho}\right)^{c}$

$$= \begin{pmatrix} \left(\left\{ <\xi, \left[{\alpha_1}^{-}{}^{\rho}_{\sigma}, {\alpha_1}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_1}^{-}{}^{\rho}_{\sigma}, {\beta_1}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right) \oplus \\ \left(\left\{ <\xi, \left[{\alpha_2}^{-}{}^{\rho}_{\sigma}, {\alpha_2}^{+}{}^{\rho}_{\sigma}(u) \right], \left[{\beta_2}^{-}{}^{\rho}_{\sigma}, {\beta_2}^{+}{}^{\rho}_{\sigma} \right] > \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \right) \end{pmatrix}^{c},$$
$$= \left(\begin{Bmatrix} <\xi, \left[{\alpha_1}^{-}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{-}{}^{\rho}_{\sigma}(\xi), {\alpha_1}^{+}{}^{\rho}_{\sigma}(\xi){\alpha_2}^{+}{}^{\rho}_{\sigma}(\xi) \right], \\ \left[{\beta_1}^{-}{}^{\rho}_{\sigma}(\xi) + {\beta_2}^{-}{}^{\rho}_{\sigma}(\xi) - {\beta_1}^{-}{}^{\rho}_{\sigma}(\xi){\beta_2}^{-}{}^{\rho}_{\sigma}(\xi), {\beta_1}^{+}{}^{\rho}_{\sigma}(\xi) + {\beta_2}^{+}{}^{\rho}_{\sigma}(\xi) - {\beta_1}^{+}{}^{\rho}_{\sigma}(\xi){\beta_2}^{+}{}^{\rho}_{\sigma}(\xi) \right] > \end{Bmatrix}, \\ \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \right)^{c},$$

$$= \left(\begin{Bmatrix} < \xi, \left[\beta_1{}^{-}{}^{\rho}_{\sigma}(\xi) + \beta_2{}^{-}{}^{\rho}_{\sigma}(\xi) - \beta_1{}^{-}{}^{\rho}_{\sigma}(\xi)\beta_2{}^{-}{}^{\rho}_{\sigma}(\xi), \beta_1{}^{+}{}^{\rho}_{\sigma}(\xi) + \beta_2{}^{+}{}^{\rho}_{\sigma}(\xi) - \beta_1{}^{+}{}^{\rho}_{\sigma}(\xi)\beta_2{}^{+}{}^{\rho}_{\sigma}(\xi)\right], \\ \left[\alpha_1{}^{-}{}^{\rho}_{\sigma}(\xi)\alpha_2{}^{-}{}^{\rho}_{\sigma}(\xi), \alpha_1{}^{+}{}^{\rho}_{\sigma}(\xi)\alpha_2{}^{+}{}^{\rho}_{\sigma}(\xi)\right] > \end{Bmatrix}, \\ 1 - \eta_1{}^{\rho}_{\sigma}(\xi)\eta_2{}^{\rho}_{\sigma}(\xi) \right),$$

$$= \left(\begin{matrix} \left(\{< \xi, [\beta_1{}^{-}{}^{\rho}_{\sigma}, \beta_1{}^{+}{}^{\rho}_{\sigma}], [\alpha_1{}^{-}{}^{\rho}_{\sigma}, \alpha_1{}^{+}{}^{\rho}_{\sigma}(u)] >\}, 1 - \eta_1{}^{\rho}_{\sigma}(\xi)\right) \oplus \\ \left(\{< \xi, [\beta_2{}^{-}{}^{\rho}_{\sigma}, \beta_2{}^{+}{}^{\rho}_{\sigma}], [\alpha_2{}^{-}{}^{\rho}_{\sigma}, \alpha_2{}^{+}{}^{\rho}_{\sigma}(u)], >\}, 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{matrix} \right),$$

$$= \left(£_{\eta_{1\sigma}}^{\rho}\right)^{c} \oplus \left(£_{\eta_{2\sigma}}^{\rho}\right)^{c}.$$

Hence proved.

The proposition that the operation of the Min-Product, Max-Product, addition, and product in GIV IF SE sets illustrate some of the properties of the total GIVIFSE set and zero GIV IF SE set.

Proposition 3. Let $B_{\eta_{\sigma}}^{\rho}$ *and* $\phi_{\eta_{\sigma}}^{\rho}$ belongs to GIVIFSE sets. Then

i. $B_{\eta_{\sigma}}^{\rho} \oplus \phi_{\eta_{\sigma}}^{\rho} = \phi_{\eta_{\sigma}}^{\rho}$
ii. $B_{\eta_{\sigma}}^{\rho} \otimes \phi_{\eta_{\sigma}}^{\rho} = \phi_{\eta_{\sigma}}^{\rho}$
iii. $B_{\eta_{\sigma}}^{\rho} \vee \phi_{\eta_{\sigma}}^{\rho} = \phi_{\eta_{\sigma}}^{\rho}$
iv. $B_{\eta_{\sigma}}^{\rho} \wedge \phi_{\eta_{\sigma}}^{\rho} = \phi_{\eta_{\sigma}}^{\rho}$

Proposition 4. Let $£_{\eta_{1\sigma}}^{\rho}$ *and* $£_{\eta_{2\sigma}}^{\rho}$ are two GIVIFSE sets. Then commutative law, involations law with respect to Max-Product and Min-Product and double negation laws also holds:

i. $£_{\eta_{1\sigma}}^{\rho} \vee £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{2\sigma}}^{\rho} \vee £_{\eta_{1\sigma}}^{\rho}$
ii. $£_{\eta_{1\sigma}}^{\rho} \wedge £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{2\sigma}}^{\rho} \wedge £_{\eta_{1\sigma}}^{\rho}$
iii. $£_{\eta_{1\sigma}}^{\rho} \oplus £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{2\sigma}}^{\rho} \oplus £_{\eta_{1\sigma}}^{\rho}$
iv. $£_{\eta_{1\sigma}}^{\rho} \otimes £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{2\sigma}}^{\rho} \otimes £_{\eta_{1\sigma}}^{\rho}$
v. $£_{\eta_{1\sigma}}^{\rho} \vee £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{1\sigma}}^{\rho}$
vi. $£_{\eta_{1\sigma}}^{\rho} \wedge £_{\eta_{2\sigma}}^{\rho} = £_{\eta_{1\sigma}}^{\rho}$
vii. $\left(\left(£_{\eta_{1\sigma}}^{\rho}\right)^{c}\right)^{c} = £_{\eta_{1\sigma}}^{\rho}$

Proof i) $£_{\eta_{1\sigma}}^{\rho} \vee £_{\eta_{2\sigma}}^{\rho}$

$$= \begin{bmatrix} \left(\left(\{< \xi, [\alpha_1{}^{-}{}^{\rho}_{\sigma}(\xi), \alpha_1{}^{+}{}^{\rho}_{\sigma}(\xi)], [\beta_1{}^{-}{}^{\rho}_{\sigma}(\xi), \beta_1{}^{+}{}^{\rho}_{\sigma}(\xi)] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) : \xi \in \hat{U}, {}^{\rho_1}_{\sigma_1} \in \hat{A}_1 \times \hat{G}_1\right) \vee \\ \left(\left(\{< \xi, [\alpha_2{}^{-}{}^{\rho}_{\sigma}(\xi), \alpha_2{}^{+}{}^{\rho}_{\sigma}(\xi)], [\beta_2{}^{-}{}^{\rho}_{\sigma}(\xi), \beta_2{}^{+}{}^{\rho}_{\sigma}(\xi)] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) : \xi \in \hat{U}, {}^{\rho_2}_{\sigma_2} \in \hat{A}_2 \times \hat{G}_2\right) \end{bmatrix},$$

$$= \begin{bmatrix} \begin{Bmatrix} < \xi, [\alpha_1{}^{-}{}^{\rho}_{\sigma}(\xi) \vee \alpha_2{}^{-}{}^{\rho}_{\sigma}(\xi), \alpha_1{}^{+}{}^{\rho}_{\sigma}(\xi) \vee \alpha_2{}^{+}{}^{\rho}_{\sigma}(\xi)], \\ [\beta_1{}^{-}{}^{\rho}_{\sigma}(\xi) \wedge \beta_2{}^{-}{}^{\rho}_{\sigma}(\xi), \beta_1{}^{+}{}^{\rho}_{\sigma}(\xi) \wedge \beta_2{}^{+}{}^{\rho}_{\sigma}(\xi)] > \end{Bmatrix}, \\ \eta_1{}^{\rho}_{\sigma}(\xi) \vee \eta_2{}^{\rho}_{\sigma}(\xi) \end{bmatrix},$$

$$= \begin{bmatrix} \begin{Bmatrix} < \xi, [\alpha_2{}^{-}{}^{\rho}_{\sigma}(\xi) \vee \alpha_1{}^{-}{}^{\rho}_{\sigma}(\xi), \alpha_2{}^{+}{}^{\rho}_{\sigma}(\xi) \vee \alpha_1{}^{+}{}^{\rho}_{\sigma}(\xi)], \\ [\beta_2{}^{-}{}^{\rho}_{\sigma}(\xi) \wedge \beta_1{}^{-}{}^{\rho}_{\sigma}(\xi), \beta_2{}^{+}{}^{\rho}_{\sigma}(\xi) \wedge \beta_1{}^{+}{}^{\rho}_{\sigma}(\xi)] > \end{Bmatrix}, \\ \eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_1{}^{\rho}_{\sigma}(\xi) \end{bmatrix},$$

$$= \begin{bmatrix} \left(\left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}(\xi), \alpha_2{}^{+\rho}_{\sigma}(\xi)], [\beta_2{}^{-\rho}_{\sigma}(\xi), \beta_2{}^{+\rho}_{\sigma}(\xi)] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) : \xi \in \hat{U}, {}^{\rho_2}_{2} \in \hat{A}_2 \times \hat{G}_2\right) \vee \\ \left(\left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}(\xi), \alpha_1{}^{+\rho}_{\sigma}(\xi)], [\beta_1{}^{-\rho}_{\sigma}(\xi), \beta_1{}^{+\rho}_{\sigma}(\xi)] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) : \xi \in \hat{U}, {}^{\rho_1}_{\sigma_1} \in \hat{A}_1 \times \hat{G}_1\right) \end{bmatrix},$$
$$= \pounds^{\rho}_{\eta_{2\sigma}} \vee \pounds^{\rho}_{\eta_{1\sigma}}.$$

4.2 More Laws of Algebra of Sets

This section examines the characteristics of GIVIFSE sets intersection and union about their operations.

Proposition 5. Let $\pounds^{\rho}_{\eta_{1\sigma}}$, $\Gamma^{\rho}_{\eta_{2\sigma}}$ and $\Upsilon^{\rho}_{\eta_{3\sigma}}$ are three GIVIFSE sets. Then

i. $\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}} = \pounds^{\rho}_{\eta_{1\sigma}} \cap \left(\Gamma^{\rho}_{\eta_{2\sigma}}\right)^c$,
ii. $\Gamma^{\rho}_{\eta_{2\sigma}} - \pounds^{\rho}_{\eta_{1\sigma}} = \Gamma^{\rho}_{\eta_{2\sigma}} \cap \left(\pounds^{\rho}_{\eta_{1\sigma}}\right)^c$,
iii. $(\pounds^{\rho}_{\eta_{1\sigma}} \cup \Gamma^{\rho}_{\eta_{2\sigma}}) - \Gamma^{\rho}_{\eta_{2\sigma}} = \pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}}$,
iv. $(\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}}) \cap \Gamma^{\rho}_{\eta_{2\sigma}} = \emptyset$,
v. $(\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}}) \cup \left(\Gamma^{\rho}_{\eta_{2\sigma}} - \pounds^{\rho}_{\eta_{1\sigma}}\right) = (\pounds^{\rho}_{\eta_{1\sigma}} \cup \Gamma^{\rho}_{\eta_{2\sigma}}) - (\pounds^{\rho}_{\eta_{1\sigma}} \cap \Gamma^{\rho}_{\eta_{2\sigma}})$,
vi. $\pounds^{\rho}_{\eta_{1\sigma}} - (\Gamma^{\rho}_{\eta_{2\sigma}} \cap \Upsilon^{\rho}_{\eta_{3\sigma}}) = (\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}}) \cup (\pounds^{\rho}_{\eta_{1\sigma}} - \Upsilon^{\rho}_{\eta_{3\sigma}})$,
vii. $\pounds^{\rho}_{\eta_{1\sigma}} - (\Gamma^{\rho}_{\eta_{2\sigma}} \cup \Upsilon^{\rho}_{\eta_{3\sigma}}) = (\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}}) \cap (\pounds^{\rho}_{\eta_{1\sigma}} - \Upsilon^{\rho}_{\eta_{3\sigma}})$,
viii. $\pounds^{\rho}_{\eta_{1\sigma}} \cap (\Gamma^{\rho}_{\eta_{2\sigma}} - \Upsilon^{\rho}_{\eta_{3\sigma}}) = (\pounds^{\rho}_{\eta_{1\sigma}} \cap \Gamma^{\rho}_{\eta_{2\sigma}}) - (\pounds^{\rho}_{\eta_{1\sigma}} \cap \Upsilon^{\rho}_{\eta_{3\sigma}})$,

Proof. (i) $\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}} = \pounds^{\rho}_{\eta_{1\sigma}} \cap \left(\Gamma^{\rho}_{\eta_{2\sigma}}\right)^c$

Let

$$\boxed{\pounds^{\rho}_{\eta_{1\sigma}} = \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right),}$$

$$\boxed{\text{and } \Gamma^{\rho}_{\eta_{2\sigma}} = \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)},$$

$$\pounds^{\rho}_{\eta_{1\sigma}} - \Gamma^{\rho}_{\eta_{2\sigma}} \boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)^c \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}], [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], >\}, 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\},\right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

Now

$$\boxed{\Gamma^{\rho}_{\eta_{2\sigma}} = \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right),}$$

$$\left(\Gamma_{\eta_{2\sigma}}^{\rho}\right)^{c}=\left(\{<\xi,[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}]>\},\eta_2{}^{\rho}_{\sigma}(\xi)\right)^{c},$$

$$=\left(\{<\xi,[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}],[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)]>\},1-\eta_2{}^{\rho}_{\sigma}(\xi)\right),$$

$$£_{\eta_{1\sigma}}^{\rho}\cap\left(\Gamma_{\eta_{2\sigma}}^{\rho}\right)^{c}=\begin{bmatrix}\left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right)\cap\\ \left(\{<\xi,[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}],[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],>\},1-\eta_2{}^{\rho}_{\sigma}(\xi)\right)\end{bmatrix},$$

$$=\begin{bmatrix}(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\beta_2{}^{+\rho}_{\sigma}],[\beta_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\vee\alpha_2{}^{+\rho}_{\sigma}]>\},)\\ \left(\eta_1{}^{\rho}_{\sigma}(\xi)\wedge 1-\eta_2{}^{\rho}_{\sigma}(\xi)\right)\end{bmatrix}.$$

Hence proved.

(ii) Straight forward.

Proof (iii) $(£_{\eta_{1\sigma}}^{\rho}\cup\Gamma_{\eta_{2\sigma}}^{\rho})-\Gamma_{\eta_{2\sigma}}^{\rho}=£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho}$

LHS

$(£_{\eta_{1\sigma}}^{\rho}\cup\Gamma_{\eta_{2\sigma}}^{\rho})-\Gamma_{\eta_{2\sigma}}^{\rho}$

$$£_{\eta_{1\sigma}}^{\rho}=\left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right),$$

$$\Gamma_{\eta_{2\sigma}}^{\rho}=\left(\{<\xi,[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}]>\},\eta_2{}^{\rho}_{\sigma}(\xi)\right)$$

$$(£_{\eta_{1\sigma}}^{\rho}\cup\Gamma_{\eta_{2\sigma}}^{\rho})=\begin{matrix}\left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right)\cup\\ \left(\{<\xi,[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}]>\},\eta_2{}^{\rho}_{\sigma}(\xi)\right)\end{matrix},$$

$$=\left(\begin{Bmatrix}<\xi,[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)],\\ [\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}]>\end{Bmatrix},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right),$$

$$(£_{\eta_{1\sigma}}^{\rho}\cup\Gamma_{\eta_{2\sigma}}^{\rho})-\Gamma_{\eta_{2\sigma}}^{\rho}=\begin{matrix}\left(\begin{Bmatrix}<\xi,[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)],\\ [\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}]>\end{Bmatrix},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right)\cap\\ \left(\left(\{<\xi,[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}]>\},\eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)^{c}\end{matrix},$$

$$=\begin{matrix}\left(\begin{Bmatrix}<\xi,[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)],\\ [\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}]>\end{Bmatrix},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right)\cap\\ \left(\{<\xi,[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}],[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)]>\},1-\eta_2{}^{\rho}_{\sigma}(\xi)\right)\end{matrix}$$

$$=\begin{matrix}\begin{Bmatrix}<\xi,[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)\wedge\beta_2{}^{+\rho}_{\sigma}],\\ [\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}\vee\alpha_2{}^{+\rho}_{\sigma}]>\end{Bmatrix},\\ \eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)1-\eta_2{}^{\rho}_{\sigma}(\xi)\end{matrix}.$$

RHS

$$£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho} \boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)^c \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}], [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], >\}, 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}}.$$

Do not hold.

Example

Let $£_{\eta_{1\sigma}}^{\rho} = (\{< \xi, [0.7,0.8], [0.6,0.9] >\}, 0.02)$

$\Gamma_{\eta_{2\sigma}}^{\rho} = (\{< \xi, [0.3,0.7], [0.5,0.8] >\}, 0.05)$

LHS $(£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) - \Gamma_{\eta_{2\sigma}}^{\rho}$

$$(£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) = \boxed{(\{< \xi, [0.7,0.8], [0.6,0.9] >\}, 0.02) \cup (\{< \xi, [0.3,0.7], [0.5,0.8] >\}, 0.05)}$$

$$= \boxed{(\{< \xi, [0.7,0.8], [0.5,0.8] >\}, 0.05)}$$

$(£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) - \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{< \xi, [0.7,0.8], [0.5,0.8] >\}, 0.05) \cap (\{< \xi, [0.3,0.7], [0.5,0.8] >\}, 0.05)^c},$$

$$\boxed{= (\{< \xi, [0.7,0.8], [0.5,0.8] >\}, 0.05) \cap (\{< \xi, [0.5,0.8], [0.3,0.7] >\}, 0.95)},$$

$$\boxed{= (\{< \xi, [0.5,0.8], [0.5,0.8] >\}, 0.05)}$$

RHS

$£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{< \xi, [0.7,0.8], [0.6,0.9] >\}, 0.02) \cap (\{< \xi, [0.3,0.7], [0.5,0.8] >\}, 0.05)^c},$$

$$\boxed{= (\{< \xi, [0.7,0.8], [0.6,0.9] >\}, 0.02) \cap (\{< \xi, [0.5,0.8], [0.3,0.7] >\}, 0.95)},$$

$$\boxed{= (\{< \xi, [0.5,0.8], [0.6,0.9] >\}, 0.02)}.$$

LHS$\neq$ ***RHS*.**

Proof (iv) $(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap \Gamma_{\eta_{2\sigma}}^{\rho} = \emptyset$

$$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) = \boxed{\begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)^c \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}], [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], >\}, 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\},\right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$(\pounds_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap \Gamma_{\eta_{2\sigma}}^{\rho}$$

$$\boxed{= \begin{bmatrix} \left\{\begin{matrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\},\right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right\} \cap \\ \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}}$$

$$= \boxed{\left(\begin{matrix} \left\{\begin{matrix} < \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma} \wedge \alpha_2{}^{+\rho}_{\sigma}], \\ [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma}] > \end{matrix}\right\}, \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{matrix} \right)}$$

$$\neq \emptyset$$

Do not hold.

Example

Let $\pounds_{\eta_{1\sigma}}^{\rho} = (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7)$,

$\Gamma_{\eta_{2\sigma}}^{\rho} = (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8)$,

$\pounds_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7) \cap (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8)^c},$$

$$\boxed{= (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7) \cap (\{< \xi, [0.5,0.7], [0.2,0.4] >\}, 0.2)^c},$$

$$\boxed{= (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.2)},$$

$(\pounds_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.2) \cap (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8)},$$

$$\boxed{= (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.2)},$$

$$\neq \emptyset.$$

Proof. (v) $(\pounds_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cup \left(\Gamma_{\eta_{2\sigma}}^{\rho} - \pounds_{\eta_{1\sigma}}^{\rho}\right) = (\pounds_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) - (\pounds_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho})$

L-H-S

Let

$$(\pounds_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) = \boxed{\begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{< \xi, [\alpha_2{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u)], [\beta_2{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)^c \end{bmatrix}},$$

$$= \left[\begin{matrix} \left(\{< \xi, [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}] >\}, {\eta_1}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}], [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], >\}, 1 - {\eta_2}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right],$$

$$= \left[\begin{matrix} (\{< \xi, [{\alpha_1}^{-\rho}_{\sigma} \wedge {\beta_2}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u) \wedge {\beta_2}^{+\rho}_{\sigma}], [{\beta_1}^{-\rho}_{\sigma} \vee {\alpha_2}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma} \vee {\alpha_2}^{+\rho}_{\sigma}] >\},) \\ \left({\eta_1}^{\rho}_{\sigma}(\xi) \wedge 1 - {\eta_2}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right],$$

$$\left(\Gamma_{\eta_{2\sigma}}^{\rho} - £_{\eta_{1\sigma}}^{\rho}\right) = \left[\begin{matrix} \left(\{< \xi, [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}] >\}, {\eta_2}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\{< \xi, [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}] >\}, {\eta_1}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right],$$

$$= \left[\begin{matrix} \left(\{< \xi, [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}] >\}, {\eta_2}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{< \xi, [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}] >\}, {\eta_1}^{\rho}_{\sigma}(\xi)\right)\right)^{c} \end{matrix}\right],$$

$$= \left[\begin{matrix} \left(\{< \xi, [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}] >\}, {\eta_2}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}], [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], >\}, 1 - {\eta_1}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right],$$

$$= \left[\begin{matrix} (\{< \xi, [{\alpha_2}^{-\rho}_{\sigma} \wedge {\beta_1}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u) \wedge {\beta_1}^{+\rho}_{\sigma}], [{\beta_2}^{-\rho}_{\sigma} \vee {\alpha_1}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma} \vee {\alpha_1}^{+\rho}_{\sigma}] >\},) \\ \left({\eta_2}^{\rho}_{\sigma}(\xi) \wedge 1 - {\eta_1}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right],$$

$$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cup \left(\Gamma_{\eta_{2\sigma}}^{\rho} - £_{\eta_{1\sigma}}^{\rho}\right)$$

$$= \begin{matrix} \left[\begin{matrix} (\{< \xi, [{\alpha_1}^{-\rho}_{\sigma} \wedge {\beta_2}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u) \wedge {\beta_2}^{+\rho}_{\sigma}], [{\beta_1}^{-\rho}_{\sigma} \vee {\alpha_2}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma} \vee {\alpha_2}^{+\rho}_{\sigma}] >\},) \\ \left({\eta_1}^{\rho}_{\sigma}(\xi) \wedge 1 - {\eta_2}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right] \cup \\ \left[\begin{matrix} (\{< \xi, [{\alpha_2}^{-\rho}_{\sigma} \wedge {\beta_1}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u) \wedge {\beta_1}^{+\rho}_{\sigma}], [{\beta_2}^{-\rho}_{\sigma} \vee {\alpha_1}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma} \vee {\alpha_1}^{+\rho}_{\sigma}] >\},) \\ \left({\eta_2}^{\rho}_{\sigma}(\xi) \wedge 1 - {\eta_1}^{\rho}_{\sigma}(\xi)\right) \end{matrix}\right] \end{matrix}$$

$$= \left[\begin{matrix} \left(\left\{\begin{matrix} < \xi, [{\alpha_1}^{-\rho}_{\sigma} \wedge {\beta_2}^{-\rho}_{\sigma} \vee {\alpha_2}^{-\rho}_{\sigma} \wedge {\beta_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u) \wedge {\beta_2}^{+\rho}_{\sigma} \vee {\alpha_2}^{+\rho}_{\sigma}(u) \wedge {\beta_1}^{+\rho}_{\sigma}], \\ [{\beta_1}^{-\rho}_{\sigma} \vee {\alpha_2}^{-\rho}_{\sigma} \wedge {\beta_2}^{-\rho}_{\sigma} \vee {\alpha_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma} \vee {\alpha_2}^{+\rho}_{\sigma} \wedge {\beta_2}^{+\rho}_{\sigma} \vee {\alpha_1}^{+\rho}_{\sigma}] > \end{matrix}\right\},\right) \\ ({\eta_1}^{\rho}_{\sigma}(\xi) \wedge (1 - {\eta_2}^{\rho}_{\sigma}(\xi)) \vee {\eta_2}^{\rho}_{\sigma}(\xi) \wedge (1 - {\eta_1}^{\rho}_{\sigma}(\xi))) \end{matrix}\right],$$

R-H-S

$$(£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) = \begin{matrix} \left(\{< \xi, [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}] >\}, {\eta_1}^{\rho}_{\sigma}(\xi)\right) \cup \\ \left(\{< \xi, [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}] >\}, {\eta_2}^{\rho}_{\sigma}(\xi)\right) \end{matrix},$$

$$= \left(\left\{\begin{matrix} < \xi, [{\alpha_1}^{-\rho}_{\sigma} \vee {\alpha_2}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u) \vee {\alpha_2}^{+\rho}_{\sigma}(u)], \\ [{\beta_1}^{-\rho}_{\sigma} \wedge {\beta_2}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma} \wedge {\beta_2}^{+\rho}_{\sigma}] > \end{matrix}\right\}, {\eta_1}^{\rho}_{\sigma}(\xi) \vee {\eta_2}^{\rho}_{\sigma}(\xi)\right),$$

$$(£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) = \begin{matrix} \left(\{< \xi, [{\alpha_1}^{-\rho}_{\sigma}, {\alpha_1}^{+\rho}_{\sigma}(u)], [{\beta_1}^{-\rho}_{\sigma}, {\beta_1}^{+\rho}_{\sigma}] >\}, {\eta_1}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\{< \xi, [{\alpha_2}^{-\rho}_{\sigma}, {\alpha_2}^{+\rho}_{\sigma}(u)], [{\beta_2}^{-\rho}_{\sigma}, {\beta_2}^{+\rho}_{\sigma}] >\}, {\eta_2}^{\rho}_{\sigma}(\xi)\right) \end{matrix},$$

$$\boxed{=\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\wedge\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\vee\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\vee\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\wedge\eta_2{}^{\rho}_{\sigma}(\xi)\right)},$$

$(£_{\eta_{1\sigma}}^{\rho}\cup\Gamma_{\eta_{2\sigma}}^{\rho})-(£_{\eta_{1\sigma}}^{\rho}\cap\Gamma_{\eta_{2\sigma}}^{\rho})$

$$\boxed{=\left(\begin{matrix}\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right)-\\ \left(\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\wedge\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\vee\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\vee\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\wedge\eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)\end{matrix}\right)},$$

$$\boxed{=\left(\begin{matrix}\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right)\cap\\ \left(\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\wedge\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\vee\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\vee\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\wedge\eta_2{}^{\rho}_{\sigma}(\xi)\right)^{c}\right)\end{matrix}\right)},$$

$$\boxed{=\left(\begin{matrix}\left(\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}\right]>\end{matrix}\right\},\eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\right)\cap\\ \left(\left(\left\{\begin{matrix}<\xi,\left[\beta_1{}^{-\rho}_{\sigma}\vee\beta_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\vee\beta_2{}^{+\rho}_{\sigma}\right],\\ ,\left[\alpha_1{}^{-\rho}_{\sigma}\wedge\alpha_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\alpha_2{}^{+\rho}_{\sigma}(u)\right]>\end{matrix}\right\},1-\eta_1{}^{\rho}_{\sigma}(\xi)\wedge\eta_2{}^{\rho}_{\sigma}(\xi)\right)\right)\end{matrix}\right)},$$

$$=\left(\begin{matrix}\left\{\begin{matrix}<\xi,\left[\alpha_1{}^{-\rho}_{\sigma}\vee\alpha_2{}^{-\rho}_{\sigma}\wedge\beta_1{}^{-\rho}_{\sigma}\vee\beta_2{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)\vee\alpha_2{}^{+\rho}_{\sigma}(u)\wedge\beta_1{}^{+\rho}_{\sigma}\vee\beta_2{}^{+\rho}_{\sigma}\right],\\ \left[\beta_1{}^{-\rho}_{\sigma}\wedge\beta_2{}^{-\rho}_{\sigma}\vee\alpha_1{}^{-\rho}_{\sigma}\wedge\alpha_2{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}\wedge\beta_2{}^{+\rho}_{\sigma}\vee\alpha_1{}^{+\rho}_{\sigma}(u)\wedge\alpha_2{}^{+\rho}_{\sigma}(u)\right]>\end{matrix}\right\},\\ \eta_1{}^{\rho}_{\sigma}(\xi)\vee\eta_2{}^{\rho}_{\sigma}(\xi)\wedge(1-\eta_1{}^{\rho}_{\sigma}(\xi))\wedge\eta_2{}^{\rho}_{\sigma}(\xi)\end{matrix}\right).$$

Do not hold.

Example

Let $£_{\eta_{1\sigma}}^{\rho}=(\{<\xi,[0.7,0.8],[0.6,0.9]>\},0.02)$,

$\Gamma_{\eta_{2\sigma}}^{\rho}=(\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)$,

$£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{=(\{<\xi,[0.7,0.8],[0.6,0.9]>\},0.02)\cap(\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)^{c}},$$

$$\boxed{=(\{<\xi,[0.3,0.5],[0.4,0.6]>\},0.7)\cap(\{<\xi,[0.5,0.8],[0.3,0.7]>\},0.95)},$$

$$\boxed{=(\{<\xi,[0.5,0.8],[0.6,0.9]>\},0.02)},$$

$\Gamma_{\eta_{2\sigma}}^{\rho}-£_{\eta_{1\sigma}}^{\rho}$

$$\boxed{=(\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)\cap(\{<\xi,[0.7,0.8],[0.6,0.9]>\},0.02)^{c}},$$

$$\boxed{=(\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)\cap(\{<\xi,[0.6,0.9],[0.7,0.8]>\},0.98)},$$

$$\boxed{=(\{<\xi,[0.3,0.7],[0.7,0.8]>\},0.05)},$$

$(£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho})\cup\Gamma_{\eta_{2\sigma}}^{\rho}-£_{\eta_{1\sigma}}^{\rho}$

$$\boxed{= (\{<\xi,[0.5,0.8],[0.6,0.9]>\},0.02) \cup (\{<\xi,[0.3,0.7],[0.7,0.8]>\},0.05)},$$

$$\boxed{= (\{<\xi,[0.5,0.8],[0.6,0.8]>\},0.05)}$$

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$£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{<\xi,[0.7,0.8],[0.6,0.9]>\},0.02) \cup (\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)},$$

$$\boxed{= (\{<\xi,[0.7,0.8],[0.5,0.8]>\},0.05)}$$

$£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}$

$$\boxed{= (\{<\xi,[0.7,0.8],[0.6,0.9]>\},0.02) \cap (\{<\xi,[0.3,0.7],[0.5,0.8]>\},0.05)},$$

$$\boxed{= (\{<\xi,[0.3,0.7],[0.6,0.9]>\},0.02)}$$

$(£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) - (£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho})$

$$\boxed{= \Big((\{<\xi,[0.7,0.8],[0.5,0.8]>\},0.05) \cap \Big((\{<\xi,[0.3,0.7],[0.6,0.9]>\},0.02)\Big)^{c}\Big)},$$

$$\boxed{= (\{<\xi,[0.7,0.8],[0.5,0.8]>\},0.05) \cap (\{<\xi,[0.6,0.9],[0.3,0.7]>\}0.98)},$$

$$\boxed{= (\{<\xi,[0.6,0.8],[0.5,0.8]>\},0.05)}.$$

L-H-S≠R-H-S

Proof .vi) $£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho}) = (£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cup (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$,

L-H-S $£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$

$$(\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho}) = \boxed{\left[\begin{array}{c} \left(\{<\xi,[\alpha_2{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u)],[\beta_2{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma}]>\},\eta_2{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left(\{<\xi,[\alpha_3{}^{-\rho}_{\sigma},\alpha_3{}^{+\rho}_{\sigma}(u)],[\beta_3{}^{-\rho}_{\sigma},\beta_3{}^{+\rho}_{\sigma}]>\},\eta_3{}^{\rho}_{\sigma}(\xi)\right)\right) \end{array}\right]},$$

$$\boxed{= \left(\left\{\begin{array}{c} <\xi,[\alpha_2{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)], \\ [\beta_2{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}]> \end{array}\right\},\eta_2{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right)},$$

$£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$

$$\boxed{= \left(\begin{array}{c} \left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\left\{\begin{array}{c} <\xi,[\alpha_2{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)], \\ [\beta_2{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}]> \end{array}\right\},\eta_2{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{array}\right)},$$

$$\boxed{= \left(\begin{array}{c} \left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left\{\begin{array}{c} <\xi,[\alpha_2{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)], \\ [\beta_2{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}]> \end{array}\right\},\eta_2{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right)^{c} \end{array}\right)},$$

$$\boxed{= \left(\begin{array}{c} \left(\{<\xi,[\alpha_1{}^{-\rho}_{\sigma},\alpha_1{}^{+\rho}_{\sigma}(u)],[\beta_1{}^{-\rho}_{\sigma},\beta_1{}^{+\rho}_{\sigma}]>\},\eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left\{\begin{array}{c} <\xi,[\beta_2{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma},\beta_2{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}] \\ ,[\alpha_2{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma},\alpha_2{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)]> \end{array}\right\},1-\eta_2{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{array}\right)},$$

$$=\left(\left\{\begin{matrix}<\xi,\left[{\alpha_1}^{-\rho}_{\sigma}\wedge{\beta_2}^{-\rho}_{\sigma}\vee{\beta_3}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}\wedge{\beta_2}^{+\rho}_{\sigma}\vee{\beta_3}^{+\rho}_{\sigma}\right]\\,\left[{\beta_1}^{-\rho}_{\sigma}\vee{\alpha_2}^{-\rho}_{\sigma}\wedge{\alpha_3}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\vee{\alpha_2}^{+\rho}_{\sigma}(u)\wedge{\alpha_3}^{+\rho}_{\sigma}(u)\right]>\end{matrix}\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\wedge 1-{\eta_2}^{\rho}_{\sigma}(\xi)\wedge{\eta_3}^{\rho}_{\sigma}(\xi)\right),$$

R-H-S $(£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho})\cup(£_{\eta_{1\sigma}}^{\rho}-\Upsilon_{\eta_{3\sigma}}^{\rho})$,

$$£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho}=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)-\\\left(\left\{<\xi,\left[{\alpha_2}^{-\rho}_{\sigma},{\alpha_2}^{+\rho}_{\sigma}(u)\right],\left[{\beta_2}^{-\rho}_{\sigma},{\beta_2}^{+\rho}_{\sigma}\right]>\right\},{\eta_2}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)\cap\\\left(\left(\left\{<\xi,\left[{\alpha_2}^{-\rho}_{\sigma},{\alpha_2}^{+\rho}_{\sigma}(u)\right],\left[{\beta_2}^{-\rho}_{\sigma},{\beta_2}^{+\rho}_{\sigma}\right]>\right\},{\eta_2}^{\rho}_{\sigma}(\xi)\right)\right)^{c}\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)\cap\\\left(\left\{<\xi,\left[{\beta_2}^{-\rho}_{\sigma},{\beta_2}^{+\rho}_{\sigma}\right],\left[{\alpha_2}^{-\rho}_{\sigma},{\alpha_2}^{+\rho}_{\sigma}(u)\right],>\right\},1-{\eta_2}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma}\wedge{\beta_2}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\wedge{\beta_2}^{+\rho}_{\sigma}\right],\left[{\beta_1}^{-\rho}_{\sigma}\vee{\alpha_2}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\vee{\alpha_2}^{+\rho}_{\sigma}\right]>\right\},\right)\\\left({\eta_1}^{\rho}_{\sigma}(\xi)\wedge 1-{\eta_2}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$$£_{\eta_{1\sigma}}^{\rho}-\Upsilon_{\eta_{3\sigma}}^{\rho}=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)-\\\left(\left\{<\xi,\left[{\alpha_3}^{-\rho}_{\sigma},{\alpha_3}^{+\rho}_{\sigma}(u)\right],\left[{\beta_3}^{-\rho}_{\sigma},{\beta_3}^{+\rho}_{\sigma}\right]>\right\},{\eta_3}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)\cap\\\left(\left(\left\{<\xi,\left[{\alpha_3}^{-\rho}_{\sigma},{\alpha_3}^{+\rho}_{\sigma}(u)\right],\left[{\beta_3}^{-\rho}_{\sigma},{\beta_3}^{+\rho}_{\sigma}\right]>\right\},{\eta_3}^{\rho}_{\sigma}(\xi)\right)\right)^{c}\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\right],\left[{\beta_1}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\right]>\right\},{\eta_1}^{\rho}_{\sigma}(\xi)\right)\cap\\\left(\left\{<\xi,\left[{\beta_3}^{-\rho}_{\sigma},{\beta_3}^{+\rho}_{\sigma}\right],\left[{\alpha_3}^{-\rho}_{\sigma},{\alpha_3}^{+\rho}_{\sigma}(u)\right],>\right\},1-{\eta_3}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$$=\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma}\wedge{\beta_3}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\wedge{\beta_3}^{+\rho}_{\sigma}\right],\left[{\beta_1}^{-\rho}_{\sigma}\vee{\alpha_3}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\vee{\alpha_3}^{+\rho}_{\sigma}\right]>\right\},\right)\\\left({\eta_1}^{\rho}_{\sigma}(\xi)\wedge(1-{\eta_3}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right],$$

$(£_{\eta_{1\sigma}}^{\rho}-\Gamma_{\eta_{2\sigma}}^{\rho})\cup(£_{\eta_{1\sigma}}^{\rho}-\Upsilon_{\eta_{3\sigma}}^{\rho})$

$$=\left(\begin{matrix}\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma}\wedge{\beta_2}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\wedge{\beta_2}^{+\rho}_{\sigma}\right],\left[{\beta_1}^{-\rho}_{\sigma}\vee{\alpha_2}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\vee{\alpha_2}^{+\rho}_{\sigma}\right]>\right\},\right)\\\left({\eta_1}^{\rho}_{\sigma}(\xi)\wedge 1-{\eta_2}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right]\cup\\\left[\begin{matrix}\left(\left\{<\xi,\left[{\alpha_1}^{-\rho}_{\sigma}\wedge{\beta_3}^{-\rho}_{\sigma},{\alpha_1}^{+\rho}_{\sigma}(u)\wedge{\beta_3}^{+\rho}_{\sigma}\right],\left[{\beta_1}^{-\rho}_{\sigma}\vee{\alpha_3}^{-\rho}_{\sigma},{\beta_1}^{+\rho}_{\sigma}\vee{\alpha_3}^{+\rho}_{\sigma}\right]>\right\},\right)\\\left({\eta_1}^{\rho}_{\sigma}(\xi)\wedge(1-{\eta_3}^{\rho}_{\sigma}(\xi)\right)\end{matrix}\right]\end{matrix}\right),$$

$$= \left[\left(\left\{ \begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\ \sigma} \wedge \beta_2{}^{-\rho}_{\ \sigma} \vee \alpha_1{}^{-\rho}_{\ \sigma} \wedge \beta_3{}^{-\rho}_{\ \sigma}, \alpha_1{}^{+\rho}_{\ \sigma}(u) \wedge \beta_2{}^{+\rho}_{\ \sigma} \vee \alpha_1{}^{+\rho}_{\ \sigma}(u) \wedge \beta_3{}^{+\rho}_{\ \sigma} \right], \\ \left[\beta_1{}^{-\rho}_{\ \sigma} \vee \alpha_2{}^{-\rho}_{\ \sigma} \wedge \beta_1{}^{-\rho}_{\ \sigma} \vee \alpha_3{}^{-\rho}_{\ \sigma}, \beta_1{}^{+\rho}_{\ \sigma} \vee \alpha_2{}^{+\rho}_{\ \sigma} \wedge \beta_1{}^{+\rho}_{\ \sigma} \vee \alpha_3{}^{+\rho}_{\ \sigma} \right] > \end{matrix} \right\}, \\ \left(\left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge (1 - \eta_2{}^{\rho}_{\sigma}(\xi)) \right) \vee \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge (1 - \eta_3{}^{\rho}_{\sigma}(\xi)) \right) \right) \right) \right].$$

Do not hold.

Example.

Let $£_{\eta_{1\sigma}}^{\rho} = (\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7)$,

$\Gamma_{\eta_{2\sigma}}^{\rho} = (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8)$,

$\Upsilon_{\eta_{3\sigma}}^{\rho} = (\{< \xi, [0.5,0.7], [0.3,0.4] >\}, 0.6)$,

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$\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho} = \boxed{\left((\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8) \cap (\{< \xi, [0.5,0.7], [0.3,0.4] >\}, 0.6) \right),}$

$\boxed{= \left((\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.6) \right),}$

$£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$

$= \boxed{\left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7) - (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.6) \right),}$

$\boxed{= \left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.4) \right)}.$

R-H-S

$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cup (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$

$£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho} = \boxed{\left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7) - (\{< \xi, [0.2,0.4], [0.5,0.7] >\}, 0.8) \right),}$

$\boxed{= \left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.2) \right)}.$

$£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho} = \boxed{\left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.7) - (\{< \xi, [0.5,0.7], [0.3,0.4] >\}, 0.6) \right),}$

$\boxed{= \left((\{< \xi, [0.3,0.4], [0.5,0.7] >\}, 0.4) \right),}$

$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cup (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$

$\boxed{= \left((\{< \xi, [0.3,0.5], [0.4,0.6] >\}, 0.2) \right) \cup \left((\{< \xi, [0.3,0.4], [0.5,0.7] >\}, 0.4) \right),}$

$\boxed{= \left((\{< \xi, [0.3,0.5], [0.5,0.7] >\}, 0.4) \right).}$

L-H-S≠R-H-S.

i. **Proof (vii).** $£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho}) = (£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$,

L-H-S $£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho})$

$$(\Gamma_{\eta_{2\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho}) = \boxed{\begin{matrix} \left(\{< \xi, [\alpha_2{}^{-\rho}_{\ \sigma}, \alpha_2{}^{+\rho}_{\ \sigma}(u)], [\beta_2{}^{-\rho}_{\ \sigma}, \beta_2{}^{+\rho}_{\ \sigma}] >\}, \eta_2{}^{\rho}_{\sigma}(\xi) \right) \cup \\ \left(\{< \xi, [\alpha_3{}^{-\rho}_{\ \sigma}, \alpha_3{}^{+\rho}_{\ \sigma}(u)], [\beta_3{}^{-\rho}_{\ \sigma}, \beta_3{}^{+\rho}_{\ \sigma}] >\}, \eta_3{}^{\rho}_{\sigma}(\xi) \right) \end{matrix}},$$

$$\boxed{= \left(\left\{ \begin{matrix} < \xi, [\alpha_2{}^{-\rho}_{\ \sigma} \vee \alpha_3{}^{-\rho}_{\ \sigma}, \alpha_2{}^{+\rho}_{\ \sigma}(u) \vee \alpha_3{}^{+\rho}_{\ \sigma}(u)], \\ [\beta_2{}^{-\rho}_{\ \sigma} \wedge \beta_3{}^{-\rho}_{\ \sigma}, \beta_2{}^{+\rho}_{\ \sigma} \wedge \beta_3{}^{+\rho}_{\ \sigma}] > \end{matrix} \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi) \right),}$$

$£_{\eta_{1\sigma}}^{\rho} - (\Gamma_{\eta_{2\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho})$

$$= \left(\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right) - \\ \left\{ \begin{array}{c} < \xi, [\alpha_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u) \vee \alpha_3{}^{+\rho}_{\sigma}(u)], \\ [\beta_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma} \wedge \beta_3{}^{+\rho}_{\sigma}] > \end{array} \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi) \end{array} \right),$$

$$= \left(\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right) \cap \\ \left(\left\{ \begin{array}{c} < \xi, [\alpha_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u) \vee \alpha_3{}^{+\rho}_{\sigma}(u)], \\ [\beta_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma} \wedge \beta_3{}^{+\rho}_{\sigma}] > \end{array} \right\}, \eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi) \right)^c \end{array} \right),$$

$$= \left(\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi) \right) \cap \\ \left(\left\{ \begin{array}{c} < \xi, [\beta_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma} \wedge \beta_3{}^{+\rho}_{\sigma}], \\ [\alpha_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u) \vee \alpha_3{}^{+\rho}_{\sigma}(u)] > \end{array} \right\}, 1 - (\eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi)) \right) \end{array} \right),$$

$$= \left(\begin{array}{c} \left(\left\{ \begin{array}{c} < \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma} \wedge \beta_3{}^{+\rho}_{\sigma}], \\ [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}(u) \vee \alpha_3{}^{+\rho}_{\sigma}(u)] > \end{array} \right\}, \right) \\ (\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - (\eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi))) \end{array} \right),$$

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$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$,

$£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}$

$$= \left[\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi) \right) \end{array} \right],$$

$£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho}$

$$= \left[\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_3{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_3{}^{+\rho}_{\sigma}] >\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_3{}^{\rho}_{\sigma}(\xi) \right) \end{array} \right],$$

$(£_{\eta_{1\sigma}}^{\rho} - \Gamma_{\eta_{2\sigma}}^{\rho}) \cap (£_{\eta_{1\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$,

$$= \begin{array}{c} \left[\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma}] >\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_2{}^{\rho}_{\sigma}(\xi) \right) \end{array} \right] \cap \\ \left[\begin{array}{c} \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_3{}^{+\rho}_{\sigma}], [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_3{}^{+\rho}_{\sigma}] >\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_3{}^{\rho}_{\sigma}(\xi) \right) \end{array} \right] \end{array},$$

$$= \left[\begin{array}{c} \left(\left\{ \begin{array}{c} < \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_2{}^{+\rho}_{\sigma} \wedge \beta_3{}^{+\rho}_{\sigma}], \\ [\beta_1{}^{-\rho}_{\sigma} \vee \alpha_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_2{}^{+\rho}_{\sigma} \vee \vee \alpha_3{}^{+\rho}_{\sigma}] > \end{array} \right\}, \right) \\ (\eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1 - (\eta_2{}^{\rho}_{\sigma}(\xi) \vee \eta_3{}^{\rho}_{\sigma}(\xi))) \end{array} \right],$$

Hence Proved.

Proof (viii). $\pounds_{\eta_{1\sigma}}^{\rho} \cap (\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho}) = (\pounds_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) - (\pounds_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$,
L-H-S $\pounds_{\eta_{1\sigma}}^{\rho} \cap (\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$
$\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho}$

$$\boxed{= \begin{bmatrix} \left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u) \wedge \beta_3{}^{+\rho}_{\sigma}\right], \left[\beta_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma} \vee \alpha_3{}^{+\rho}_{\sigma}\right] >\right\}, \right) \\ \left(\eta_2{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$\pounds_{\eta_{1\sigma}}^{\rho} \cap (\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$

$$\boxed{= \begin{bmatrix} \left(\left\{< \xi, \left[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\right], \left[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\right] >\right\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left\{< \xi, \left[\alpha_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_2{}^{+\rho}_{\sigma}(u) \wedge \beta_3{}^{+\rho}_{\sigma}\right], \left[\beta_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_2{}^{+\rho}_{\sigma} \vee \alpha_3{}^{+\rho}_{\sigma}\right] >\right\}, \right) \\ \left(\eta_2{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma} \wedge \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma} \wedge \alpha_2{}^{+\rho}_{\sigma}(u) \wedge \beta_3{}^{+\rho}_{\sigma}\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma} \vee \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma} \vee \alpha_3{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi) \wedge 1 - \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}}.$$

R-H-S $(\pounds_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) - (\pounds_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$,

$$\boxed{= \begin{bmatrix} \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_2{}^{+\rho}_{\sigma}(u)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi)\right) - \\ \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_2{}^{+\rho}_{\sigma}(u)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi)\right)^{c} \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_2{}^{+\rho}_{\sigma}(u)\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi)\right) \cap \\ \left(\left\{\begin{matrix} < \xi, \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}\right], \\ \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}(u)\right] > \end{matrix}\right\}, 1 - (\eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi))\right) \end{bmatrix}},$$

$$\boxed{= \begin{bmatrix} \left(\left\{\begin{matrix} < \xi, \left[\alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_2{}^{-\rho}_{\sigma} \wedge \beta_1{}^{-\rho}_{\sigma} \vee \beta_3{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma} \wedge \alpha_2{}^{+\rho}_{\sigma}(u) \wedge \beta_1{}^{+\rho}_{\sigma} \vee \beta_3{}^{+\rho}_{\sigma}\right], \\ \left[\beta_1{}^{-\rho}_{\sigma} \vee \beta_2{}^{-\rho}_{\sigma} \vee \alpha_1{}^{-\rho}_{\sigma} \wedge \alpha_3{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \beta_2{}^{+\rho}_{\sigma} \vee \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \alpha_3{}^{+\rho}_{\sigma}\right] > \end{matrix}\right\}, \right) \\ \left(\eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_2{}^{\rho}_{\sigma}(\xi) \wedge 1 - ((\eta_1{}^{\rho}_{\sigma}(\xi) \wedge \eta_3{}^{\rho}_{\sigma}(\xi))\right) \end{bmatrix}}.$$

Do not hold.

Example

Let $£_{\eta_{1\sigma}}^{\rho} = (\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3)$,
$\Gamma_{\eta_{2\sigma}}^{\rho} = (\{<\xi, [0.4,0.5], [0.5,0.7]>\}, 0.1)$,
$\Upsilon_{\eta_{3\sigma}}^{\rho} = (\{<\xi, [0.6,0.7], [0.3,0.4]>\}, 0.4)$,
L-H-S

$$\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho} \boxed{= \Big((\{<\xi, [0.4,0.5], [0.5,0.7]>\}, 0.1) - \big((\{<\xi, [0.6,0.7], [0.3,0.4]>\}, 0.4)\big)\Big),}$$
$$\boxed{= \Big((\{<\xi, [0.4,0.5], [0.5,0.7]>\}, 0.1) \cap \big((\{<\xi, [0.6,0.7], [0.3,0.4]>\}, 0.4)\big)^{c}\Big),}$$
$$\boxed{= \Big((\{<\xi, [0.4,0.5], [0.5,0.7]>\}, 0.1) \cap \big((\{<\xi, [0.3,0.4], [0.6,0.7]>\}, 0.6)\big)\Big),}$$
$$\boxed{= \big((\{<\xi, [0.3,0.4], [0.6,0.7]>\}, 0.1)\big),}$$

$£_{\eta_{1\sigma}}^{\rho} \cap (\Gamma_{\eta_{2\sigma}}^{\rho} - \Upsilon_{\eta_{3\sigma}}^{\rho})$
$$= \boxed{\Big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3) \cap \big((\{<\xi, [0.3,0.4], [0.6,0.7]>\}, 0.1)\big)\Big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.6,0.7]>\}, 0.1)\big)}.$$

R-H-S $(£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) - (£_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$,

$$£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho} \boxed{= \Big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3) \cap \big((\{<\xi, [0.4,0.5], [0.5,0.7]>\}, 0.1)\big)\Big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.5,0.7]>\}, 0.1)\big),}$$
$$£_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho} \boxed{= \big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3) \cap (\{<\xi, [0.6,0.7], [0.3,0.4]>\}, 0.4)\big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3)\big),}$$

$(£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) - (£_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.5,0.7]>\}, 0.1) - (\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3)\big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.5,0.7]>\}, 0.1) \cap (\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3)^{c}\big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.5,0.7]>\}, 0.1) \cap (\{<\xi, [0.3,0.6], [0.2,0.3]>\}, 0.7)\big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.5,0.7]>\}, 0.1)\big).}$$

L-H-S≠R-H-S.

4.3 The Fundamental Laws of GIVIFSE sets Of Algebra

In this section we check some properties of complement law, distributive law, identity law and complement law for total and zero of GIVIFSE sets with respect to their operations.

Proposition 6. For any GIVIFSE sets $£_{\eta_{1\sigma}}^{\rho}$, $\Gamma_{\eta_{2\sigma}}^{\rho}$ and $\Upsilon_{\eta_{3\sigma}}^{\rho}$ the following identities hold:

Distributive law:

i. $£_{\eta_{1\sigma}}^{\rho} \cup \left(\Gamma_{\eta_{2\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho}\right) = (£_{\eta_{1\sigma}}^{\rho} \cup \Gamma_{\eta_{2\sigma}}^{\rho}) \cap (£_{\eta_{1\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho})$

ii. $£_{\eta_{1\sigma}}^{\rho} \cap \left(\Gamma_{\eta_{2\sigma}}^{\rho} \cup \Upsilon_{\eta_{3\sigma}}^{\rho}\right) = (£_{\eta_{1\sigma}}^{\rho} \cap \Gamma_{\eta_{2\sigma}}^{\rho}) \cup (£_{\eta_{1\sigma}}^{\rho} \cap \Upsilon_{\eta_{3\sigma}}^{\rho})$

Proof. Straight forward.

Proposition 7. For any GIVIFSE subsets $£_{\eta_{1\sigma}}^{\rho}$ and total GIVIFSE set $\mathrm{B}_{\eta_{\sigma}}^{\rho}$ then the following identities hold:

Identity law:

i. $£_{\eta_{1\sigma}}^{\rho} \cup \phi_{\eta_{\sigma}}^{\rho} = £_{\eta_{1\sigma}}^{\rho}$

ii. $£_{\eta_{1\sigma}}^{\rho} \cap B_{\eta_{\sigma}}^{\rho} = £_{\eta_{1\sigma}}^{\rho}$

Proof (i).

L-H-S $£_{\eta_{1\sigma}}^{\rho} \cup \phi_{\eta_{\sigma}}^{\rho}$

$$= \boxed{\left(\begin{array}{c}\left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cup \\ ((\{< \xi, [0,0], [1,1] >\}, 0))\end{array}\right)},$$

$$\boxed{= \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \vee 0, \alpha_1{}^{+\rho}_{\sigma}(u) \vee 0], [\beta_1{}^{-\rho}_{\sigma} \wedge 1, \beta_1{}^{+\rho}_{\sigma} \wedge 1] >\}, \eta_1{}^{\rho}_{\sigma}(\xi) \vee 0\right)},$$

$$\boxed{= \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)},$$

$$= £_{\eta_{1\sigma}}^{\rho}.$$

=R-H-S.

Proof (ii). $£_{\eta_{1\sigma}}^{\rho} \cap B_{\eta_{\sigma}}^{\rho} = £_{\eta_{1\sigma}}^{\rho}$

L-H-S $£_{\eta_{1\sigma}}^{\rho} \cap B_{\eta_{\sigma}}^{\rho}$

$$= \boxed{\left(\begin{array}{c}\left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cap \\ ((\{< \xi, [1,1], [0,0] >\}, 1))\end{array}\right)},$$

$$\boxed{= \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma} \wedge 1, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge 1], [\beta_1{}^{-\rho}_{\sigma} \vee 0, \beta_1{}^{+\rho}_{\sigma} \vee 0] >\}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge 1\right)},$$

$$\boxed{= \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)},$$

$$= £_{\eta_{1\sigma}}^{\rho}.$$

Complement law:

i. $£_{\eta_{1\sigma}}^{\rho} \cup £_{\eta_{1\sigma}}^{\rho\ c} = B_{\eta_{\sigma}}^{\rho}$

ii. $£_{\eta_{1\sigma}}^{\rho} \cap £_{\eta_{1\sigma}}^{\rho\ c} = \phi_{\eta_{\sigma}}^{\rho}$

Proof i). $£_{\eta_{1\sigma}}^{\rho} \cup £_{\eta_{1\sigma}}^{\rho\ c} = B_{\eta_{\sigma}}^{\rho}$

Let $£_{\eta_{1\sigma}}^{\rho}$ $\boxed{= \left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right)}$,

$£_{\eta_{1\sigma}}^{\rho\ c}$ $\boxed{= \left(\{< \xi, [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}], [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], >\}, 1 - \eta_1{}^{\rho}_{\sigma}(\xi)\right)}$,

$$£_{\eta_{1\sigma}}^{\rho} \cup £_{\eta_{1\sigma}}^{\rho\ c} = \boxed{\begin{array}{c}\left(\{< \xi, [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}] >\}, \eta_1{}^{\rho}_{\sigma}(\xi)\right) \cup \\ \left(\{< \xi, [\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}], [\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)], >\}, 1 - \eta_1{}^{\rho}_{\sigma}(\xi)\right)\end{array}},$$

$$\boxed{= \left(\left\{\begin{array}{c}< \xi, [\alpha_1{}^{-\rho}_{\sigma} \vee \beta_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \vee \beta_1{}^{+\rho}_{\sigma}], \\ [\beta_1{}^{-\rho}_{\sigma} \wedge \alpha_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \wedge \alpha_1{}^{+\rho}_{\sigma}(u)] >\end{array}\right\}, \eta_1{}^{\rho}_{\sigma}(\xi) \vee (1 - \eta_1{}^{\rho}_{\sigma}(\xi))\right)},$$

$$\neq B_{\eta_{\sigma}}^{\rho}.$$

Example

Let $£_{\eta_{1\sigma}}^{\rho} = (\{< \xi, [0.2,0.3], [0.3,0.6] >\}, 0.3)$,

$£_{\eta_{1\sigma}}^{\rho\ c} = (\{< \xi, [0.3,0.6], [0.2,0.3] >\}, 0.7)$,

L-H-S

$$£_{\eta_{1\sigma}}^{\rho} \cup £_{\eta_{1\sigma}}^{\rho\ c}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3) \cup (\{<\xi, [0.3,0.6], [0.2,0.3]>\}, 0.7)\big),}$$
$$\boxed{= \big((\{<\xi, [0.3,0.6], [0.2,0.3]>\}, 0.7)\big),}$$
$$\boxed{\neq \big((\{<\xi, [1,1], [0,0]>\}, 1)\big).}$$

L-H-S≠R-H-S

Proof ii). $£_{\eta_{1\sigma}}^{\rho} \cap £_{\eta_{1\sigma}}^{\rho\ c} = \phi_{\eta_{\sigma}}^{\rho}$

Let $£_{\eta_{1\sigma}}^{\rho} \boxed{= \Big(\big\{<\xi, \big[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\big], \big[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\big]>\big\}, \eta_1{}^{\rho}_{\sigma}(\xi)\Big)},$

$£_{\eta_{1\sigma}}^{\rho\ c} \boxed{= \Big(\big\{<\xi, \big[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\big], \big[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\big], >\big\}, 1-\eta_1{}^{\rho}_{\sigma}(\xi)\Big)},$

$$£_{\eta_{1\sigma}}^{\rho} \cap £_{\eta_{1\sigma}}^{\rho\ c} = \boxed{\begin{matrix}\Big(\big\{<\xi, \big[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\big], \big[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\big]>\big\}, \eta_1{}^{\rho}_{\sigma}(\xi)\Big) \cap \\ \Big(\big\{<\xi, \big[\beta_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma}\big], \big[\alpha_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u)\big], >\big\}, 1-\eta_1{}^{\rho}_{\sigma}(\xi)\Big)\end{matrix}},$$
$$\boxed{= \left(\begin{Bmatrix}<\xi, \big[\alpha_1{}^{-\rho}_{\sigma} \wedge \beta_1{}^{-\rho}_{\sigma}, \alpha_1{}^{+\rho}_{\sigma}(u) \wedge \beta_1{}^{+\rho}_{\sigma}\big], \\ \big[\beta_1{}^{-\rho}_{\sigma} \vee \alpha_1{}^{-\rho}_{\sigma}, \beta_1{}^{+\rho}_{\sigma} \vee \alpha_1{}^{+\rho}_{\sigma}(u)\big]>\end{Bmatrix}, \eta_1{}^{\rho}_{\sigma}(\xi) \wedge (1-\eta_1{}^{\rho}_{\sigma}(\xi))\right)},$$
$$\neq \phi_{\eta_{\sigma}}^{\rho}.$$

Example

Let $£_{\eta_{1\sigma}}^{\rho} = (\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3),$

$£_{\eta_{1\sigma}}^{\rho\ c} = (\{<\xi, [0.3,0.6], [0.2,0.3]>\}, 0.7),$

L-H-S

$$£_{\eta_{1\sigma}}^{\rho} \cap £_{\eta_{1\sigma}}^{\rho\ c}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3) \cap (\{<\xi, [0.3,0.6], [0.2,0.3]>\}, 0.7)\big),}$$
$$\boxed{= \big((\{<\xi, [0.2,0.3], [0.3,0.6]>\}, 0.3)\big),}$$
$$\boxed{\neq \big((\{<\xi, [0,0], [1,1]>\}, 0)\big).}$$

L-H-S≠R-H-S

Complement law for total and zero GIVIFSE sets

i. $B_{\eta_{\sigma}}^{\rho c} = \phi_{\eta_{\sigma}}^{\rho},$

ii. $\phi_{\eta_{\sigma}}^{\rho c} = B_{\eta_{\sigma}}^{\rho}.$

Proof i).

Let $B_{\eta_{\sigma}}^{\rho} = \boxed{\big((\{<\xi, [1,1], [0,0]>\}, 1)\big),}$

$$B_{\eta_{\sigma}}^{\rho c} = \boxed{\big((\{<\xi, [0,0], [1,1]>\}, 1-1)\big),}$$
$$B_{\eta_{\sigma}}^{\rho c} = \boxed{\big((\{<\xi, [0,0], [1,1]>\}, 0)\big),}$$

$=\phi_{\eta_{\sigma}}^{\rho}.$

Proof ii).

Let $\phi_{\eta_{\sigma}}^{\rho} = \boxed{\big((\{<\xi, [0,0], [1,1]>\}, 0)\big),}$

$$\phi_{\eta_{\sigma}}^{\rho c} = \boxed{\big((\{<\xi, [1,1], [0,0]>\}, 1-0)\big),}$$

$\phi_{\eta_{\sigma}}^{\rho c} = \boxed{((\{<\xi, [1,1], [0,0]>\}, 1)),}$ $= B_{\eta_{\sigma}}^{\rho}$.

4.4 The Algebra of Relative Complements

1. The concept of algebra of relative complement plays a vital role in the study of set-theoretic operations within the framework of GIVIFSE sets.
2. It provides a mathematical structure to handle uncertainty, and expert opinion by extending classical set operations to fuzzy and soft environments.
3. The relative complement operation in GIVIFSE sets describes how one fuzzy soft expert set differs from another, allowing comparative analysis between expert evaluations under Interval-Valued Intuitionistic conditions.
4. This algebraic operation helps in defining logical difference, exclusion relation, and complementary reasoning among GFSE sets.
5. It is useful for decision-making, pattern recognition, medical diagnosis, and information analysis in uncertain environments.

5. Conclusions

In this paper, we introduced the concept of Generalized Interval-Valued Intuitionistic Fuzzy Soft Expert Sets (GIVIFSESs) as a natural extension of generalized interval-valued fuzzy soft sets within the framework of interval-valued intuitionistic fuzzy soft expert sets. The proposed model enhances the expressive capability of existing soft expert frameworks by incorporating generalized parameterization together with interval-valued intuitionistic fuzzy information, thereby providing a more comprehensive mathematical representation of uncertainty, vagueness, and expert-dependent knowledge. This generalization significantly enriches the theoretical foundation of fuzzy soft expert systems and offers greater flexibility for representing complex uncertain information. To establish a rigorous mathematical basis for the proposed model, a comprehensive algebraic framework of GIVIFSESs was developed by introducing several fundamental operations, including sum, product, subtraction, max-product, min-product, complement, power, and scalar product. The algebraic behavior of these operations was systematically investigated, and various fundamental algebraic laws, inclusion relations, and relative complement properties were established. These theoretical results provide a consistent and unified algebraic structure for generalized interval-valued intuitionistic fuzzy soft expert sets, extending several existing concepts in fuzzy soft set theory and creating a solid mathematical foundation for further developments in generalized fuzzy algebra. Consequently, the proposed framework contributes not only to the theoretical advancement of soft expert systems but also to the broader development of algebraic approaches for modeling uncertainty.

The proposed algebraic framework also provides a promising foundation for future research. One natural direction is the development of new similarity and distance measures for GIVIFSESs, extending existing studies on generalized interval-valued intuitionistic fuzzy soft expert sets and their applications in medical diagnosis and intelligent decision support [24,33]. Furthermore, the established algebraic properties can be employed to construct novel aggregation operators, ranking methods, and multi-attribute decision-making algorithms under generalized fuzzy environments, building upon recent advances in aggregation theory, hesitant fuzzy decision models, and Fermatean fuzzy optimization techniques [27,30,34–36]. Another interesting research direction is the integration of the proposed framework with generalized rough set models, fuzzy graph theory, and algebraic fuzzy structures to develop richer mathematical models for uncertainty analysis and intelligent information processing [16,28,31,32,37]. These theoretical extensions are expected to further strengthen the mathematical foundation of generalized fuzzy systems while broadening the applicability of GIVIFSESs in complex expert-oriented environments.

Beyond theoretical developments, the proposed framework has the potential to support emerging intelligent systems that require reliable uncertainty modeling and expert reasoning. In particular, its integration with explainable artificial intelligence, advanced decision-support systems, and sustainable optimization methodologies offers promising opportunities for future research. The mathematical structure developed in this paper can serve as a foundation for designing robust fuzzy decision-making frameworks in healthcare, engineering, intelligent information processing, environmental sustainability, and socio-economic planning, where uncertainty, expert knowledge, and parameterized information must be handled simultaneously [38–45]. Therefore, we believe that the proposed algebraic framework will stimulate further theoretical investigations and inspire new applications of generalized interval-valued intuitionistic fuzzy soft expert sets across both mathematical research and intelligent computational systems.

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