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α-Domination in Fuzzy Graphs with Applications to Fake News Control

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ABSTRACT

The uncontrolled spread of misinformation across online social networks poses serious societal risks, including political manipulation, public health misinformation and erosion of public trust. Due to uncertainty in user interactions and varying influence levels, classical crisp graph models are often inadequate for realistic modeling. In this paper, we introduce an α-domination framework in fuzzy graphs to model and control fake news propagation. We formally define fuzzy coverage via direct and two-step influence with decay and establish theoretical properties of the α-domination number, including upper bounds, monotonicity and results for complete fuzzy graphs. we demonstrate how strategically selected α-dominating sets can act as fact-checking nodes to effectively reduce misinformation spread. Our framework provides a mathematically rigorous and flexible tool for misinformation containment under uncertainty, bridging fuzzy graph theory and social network analysis.

1. Introduction

The exponential growth of online social networking platforms has fundamentally transformed the way information is produced, shared and consumed. While these platforms facilitate global communication, they enable the rapid spread of misinformation and fake news, leading to public panic, political manipulation and erosion of trust. Classical graph models represent users as vertices and relationships as edges with crisp (binary) values. However, real-world social interactions are inherently uncertain and graded: trust levels vary among users, interactions may be intermittent and influence strengths differ from one connection to another. These limitations motivate the use of fuzzy graphs, where both vertices and edges are assigned membership values that represent the degree of presence or strength.

The foundation of fuzzy graphs goes back to Azriel Rosenfeld, who in 1975 extended classical graph-theoretic concepts to the fuzzy-graph setting based on fuzzy relations. The first definition of fuzzy

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graph given by Kauffman [5] in 1973, was based on fuzzy relations presented by L. Zadeh [16]. Building on Kauffman's idea, Rosenfeld (1975)[11] introduced the concept of fuzzy graphs by providing a comprehensive definition and establishing several foundational results as fuzzy analogs of classical graph theory. Domination theory in graphs has been widely used in network control, monitoring and optimization problems. A book on the subject of domination of graphs [4] lists over 1200 papers related to domination of graphs and several thousand articles on the topic have appeared since the publication of the book. Nevertheless, most existing domination frameworks are based on crisp graphs and fail to accommodate uncertainty in relationships. The study of domination in fuzzy graphs began (in a systematic way) with A. Somasundaram and S. Somasundaram, who in 1998 introduced definitions of dominating set, total dominating set, minimum dominating set and domination number for fuzzy graphs[13]. A recent survey on the domination of fuzzy graphs is available in [10].

Although fuzzy domination has been studied, the concept of α-domination incorporating multi-step influence in fuzzy networks remains underexplored, especially in the context of misinformation containment. Research on misinformation containment often uses traditional graph-theoretic models, such as the Susceptible-Infected (SI) epidemic model, or heuristic algorithms that balance the propagation of positive information and the containment of negative information. Still, these generally do not explicitly employ the mathematical framework of fuzzy domination. This represents a significant gap, as social influence is rarely binary: trust between users can vary in degree, susceptibility to false information can be topic dependent and influence may decay with distance. By addressing this gap, the present work contributes a novel framework that combines fuzzy domination theory with misinformation spread modeling that captures the uncertainty in social influence while optimizing intervention placement. Unlike existing domination models in fuzzy graphs, the proposed framework explicitly incorporates multi-step influence with decay and provides a coverage-based formulation that aligns naturally with information propagation processes in social networks. The main contributions of this paper are:

1. A novel definition of α-domination in fuzzy graphs incorporating direct and two-step influence with decay.
2. Theoretical analysis including bounds, monotonicity and exact results for special fuzzy graph classes.
3. Algorithmic computation of α-dominating sets.
4. Application to fake news control in social networks.

2. Preliminaries

Definition 1 (Fuzzy Graph) *A fuzzy graph is an ordered quadruple*

$$G = (V, E, \mu, \omega),$$

where V is a non-empty finite set of vertices, $E \subseteq V \times V$, $\mu : V \to [0, 1]$ is the vertex membership function and $\omega : V \times V \to [0, 1]$ is the edge membership function.

We model a social network as a fuzzy graph where:

- Nodes are users with activity levels $\mu(v)$.
- Edges are weighted by influence strengths $\omega(u, v)$.

- Fake news spreads along edges with probability proportional to ω.
- An α-dominating set S is selected as monitors to inject fact-checking.

In misinformation contexts, influence is not binary. A user with higher credibility and stronger connections is more likely to affect the opinions of others. Unless stated otherwise, the fuzzy graph is assumed to be directed, with $\omega(u, v)$ representing the influence of user u on user v.

2.1 α-Domination in Fuzzy Graphs

Definition 2 (Coverage Function) *Let $G = (V, E, \mu, \omega)$ be a fuzzy graph and let $S \subseteq V$. For any vertex $v \in V$, the coverage of v by S is defined as*

$$\text{cov}_S(v) = \max\left\{\mu(v).\mathbf{1}_{\{v\in S\}}, \max_{u\in S}\big(\mu(v).\,\omega(u,v)\big), \lambda.\max_{u\in S,\, w\in N(v)}\big(\mu(v).\,\omega(u,w).\,\omega(w,v)\big)\right\},$$

where $N(v) = \{x \in V : \omega(x, v) > 0\}$ and $0 < \lambda \leq 1$ is a decay factor.

Note that coverage is computed for each vertex v, but with respect to a set S. The first term is the direct membership, the second term is the direct neighbor influence and the third term is the two-step influence. Therefore, it It aggregates self-membership, direct influence from vertices in S and two-step influence from vertices in S.

Remark 1 *The intermediate vertex w in the two-step influence term is restricted to the fuzzy neighborhood of v to ensure that only valid length-two influence paths are considered.*

Lemma 1 *For a fixed vertex $v \in V$, the coverage function $\text{cov}_S(v)$ is monotone non-decreasing with respect to set inclusion. That is, if $S \subseteq T \subseteq V$, then*

$$\text{cov}_S(v) \leq \text{cov}_T(v).$$

Proof: Fix a vertex $v \in V$ and let $S \subseteq T \subseteq V$. By definition,

$$\text{cov}_S(v) = \max\left\{\mu(v)\mathbf{1}_{\{v\in S\}}, \max_{u\in S}\big(\mu(v)\omega(u,v)\big), \lambda\max_{u\in S,\, w\in N(v)}\big(\mu(v)\omega(u,w)\omega(w,v)\big)\right\}.$$

Since $S \subseteq T$, the following hold:

- $\mathbf{1}_{\{v\in S\}} \leq \mathbf{1}_{\{v\in T\}}$,
- $\max_{u\in S} \mu(v)\omega(u, v) \leq \max_{u\in T} \mu(v)\omega(u, v)$,
- $\max_{u\in S,\, w\in N(v)} \mu(v)\omega(u, w)\omega(w, v) \leq \max_{u\in T,\, w\in N(v)} \mu(v)\omega(u, w)\omega(w, v)$.

Multiplication by the constant decay factor $\lambda > 0$ preserves the inequality. Therefore, each of the three terms defining $\text{cov}_S(v)$ is less than or equal to the corresponding term defining $\text{cov}_T(v)$.

Taking the maximum of the three terms in each case yields

$$\text{cov}_S(v) \leq \text{cov}_T(v).$$

Hence, the coverage function is monotone non-decreasing with respect to set inclusion.

Definition 3 (α-Dominating Set) *Let $\alpha \in (0, 1]$. A set $S \subseteq V$ is an α-dominating set if*

$$\mathrm{cov}_S(v) \geq \alpha.\mu(v) \quad \textit{for all } v \in V.$$

where $\alpha \in (0, 1]$ is a fixed threshold.

Definition 4 (α-Domination Number) *The α-domination number of G, denoted by $\gamma_\alpha(G)$, is the minimum cardinality of an α-dominating set.*

Remark 2 *The coverage function $\mathrm{cov}_S(v)$ is defined for each vertex $v \in V$ with respect to a set $S \subseteq V$. There is no single numerical coverage associated with a set; rather, domination is verified by ensuring that every vertex individually satisfies the α-coverage condition.*

3. Greedy Algorithm

A greedy algorithm is proposed to compute an α-dominating set by iteratively selecting vertices that maximize marginal coverage. The algorithm terminates in at most $|V|$ iterations since coverage values are monotone and bounded.

3.1 Greedy Algorithm for α-Dominating Set

Algorithm
Input: Fuzzy graph $G = (V, E, \mu, \omega)$, threshold $\alpha \in (0, 1]$, decay factor $\lambda \in (0, 1]$
Output: An α-dominating set $D \subseteq V$
Initialize $D \leftarrow \emptyset$
FOR each vertex $v \in V$
Set $\mathrm{cov}(v) \leftarrow 0$
ENDFOR
WHILE there exists $v \in V$ such that $\mathrm{cov}(v) < \alpha\,\mu(v)$
Select a vertex $u \in V \setminus D$ that maximizes the marginal increase

$$\sum_{v \in V} \min(\alpha\mu(v) - \mathrm{cov}(v),\ \Delta\mathrm{cov}_u(v)),$$

where

$$\Delta\mathrm{cov}_u(v) = \max\left\{\mu(v)\omega(u, v),\ \lambda \max_{w \in N(v)} \big(\mu(v)\omega(u, w)\omega(w, v)\big)\right\}.$$

Update $D \leftarrow D \cup \{u\}$
FOR each vertex $v \in V$
Update

$$\mathrm{cov}(v) \leftarrow \max\big(\mathrm{cov}(v), \Delta\mathrm{cov}_u(v)\big)$$

ENDFOR
ENDWHILE
RETURN D

Theorem 1 *The greedy α-dominating set algorithm terminates in at most $|V|$ iterations.*

Proof: At each iteration, a new vertex is added to the set D. Coverage values $\mathrm{cov}(v)$ are monotonically non-decreasing and bounded above by $\mu(v)$. Since at least one previously under-covered vertex strictly increases its coverage at each iteration, the loop cannot continue indefinitely. As no vertex is added more than once, the algorithm terminates after at most $|V|$ iterations.

Remark 3 *The greedy algorithm runs in polynomial time, with worst-case complexity $O(|V|^3)$ due to coverage updates. While optimality is not guaranteed, the algorithm provides an efficient heuristic suitable for large networks.*

3.2 Illustrative Example: Computation of an α-Dominating Set

In this section, we present a detailed example to illustrate the computation of an α-dominating set in a fuzzy graph using the proposed coverage function and algorithm.

Example 1 *Consider the fuzzy graph*

$$G = (V, E, \mu, \omega),$$

where

$$V = \{v_1, v_2, v_3, v_4\}.$$

Assume uniform vertex memberships:

$$\mu(v_i) = 1 \quad \textit{for all } i = 1, 2, 3, 4.$$

The fuzzy edge memberships are given by:

$$\omega(v_1, v_2) = 0.8, \quad \omega(v_2, v_3) = 0.7, \quad \omega(v_3, v_4) = 0.6, \quad \omega(v_1, v_3) = 0.4.$$

Let the domination parameters be

$$\alpha = 0.5 \quad \textit{and} \quad \lambda = 0.8.$$

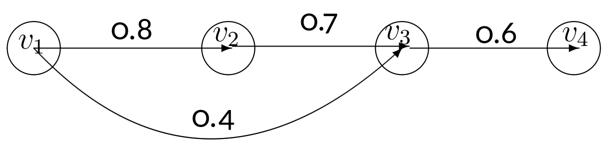

Figure 1: Fuzzy graph used for computing an α-dominating set

Step-by-Step Computation:

Initially, let

$$D = \emptyset, \qquad \mathrm{cov}(v_i) = 0 \text{ for all } i.$$

Step 1: Selection of v_1

The coverage contributions of the vertex v_1 are computed as follows:

- Self-coverage:

$$\mathrm{cov}(v_1) = \mu(v_1) = 1.$$

- Direct coverage of v_2:

$$\mathrm{cov}(v_2) = \mu(v_2)\omega(v_1, v_2) = 0.8.$$

- Direct coverage of v_3:
$$\mathrm{cov}(v_3) = \mu(v_3)\omega(v_1, v_3) = 0.4.$$
- Two-step coverage of v_4 via v_3:
$$\mathrm{cov}(v_4) = \lambda\mu(v_4)\omega(v_1, v_3)\omega(v_3, v_4) = 0.8 \times 0.4 \times 0.6 = 0.192.$$

The coverage status after selecting v_1 is shown in Table 1.

Vertex	Coverage	Required $(\alpha\mu)$
v_1	1.000	0.5
v_2	0.800	0.5
v_3	0.400	0.5
v_4	0.192	0.5

Table 1: Coverage after selecting v_1

Thus,
$$D = \{v_1\}.$$

Step 2: Selection of v_3

Vertex v_3 is added to increase the coverage of under-covered vertices.

- Self-coverage:
$$\mathrm{cov}(v_3) = 1.$$
- Direct coverage of v_4:
$$\mathrm{cov}(v_4) = \max(0.192,\ 0.6) = 0.6.$$

The updated coverage values are shown in Table 2.

Vertex	Coverage	Required $(\alpha\mu)$
v_1	1.000	0.5
v_2	0.800	0.5
v_3	1.000	0.5
v_4	0.600	0.5

Table 2: Coverage after selecting v_3

Result

Since all vertices now satisfy
$$\mathrm{cov}(v_i) \geq \alpha\mu(v_i),$$
the set
$$D = \{v_1, v_3\}$$
is an α-dominating set of the fuzzy graph. Hence,
$$\gamma_{0.5}(G) = 2.$$

This example demonstrates how the proposed greedy algorithm selects dominant vertices based on maximal marginal coverage.

Remark 4 *This example demonstrates that domination in fuzzy graphs is verified vertex-wise. A vertex may be dominated either through direct fuzzy influence or via a valid two-step influence path with decay.*

4. Theoretical Results

This section establishes bounds, monotonicity properties and exact domination numbers for complete fuzzy graphs and fuzzy paths, reduction to classical domination, Nordhaus-Gaddum type inequalities, fuzzy domatic bounds and stability under edge strength perturbations.

Theorem 2 *(Upper bound on coverage; necessary condition for existence)*
For every $v \in V$ and every $S \subseteq V$ we have

$$\text{cov}_S(v) \leq \mu(v).$$

Consequently, an α-dominating set can exist only if

$$\min_{v \in V} \mu(v) \geq \alpha.$$

Proof: Every term inside the outer max defining $\text{cov}_S(v)$ is of the form $\mu(v) \times$ (something in $[0, 1]$).
Therefore, all the three terms inside the outer max defining $\text{cov}_S(v)$ are less or equal to $\mu(v)$ because $\lambda, \omega(u, v) \leq 1$. Taking the maximum of these three quantities cannot exceed $\mu(v)$. So

$$\text{cov}_S(v) \leq \mu(v).$$

If for some $v \in V$, we have $\mu(v) < \alpha$ then $\text{cov}_S(v) \leq \mu(v) < \alpha$ for every subset S of V. Therefore, no α-dominating set can exist. hence the theorem follows.

Theorem 3 *(Monotonicity): If $0 < \alpha_1 < \alpha_2 \leq 1$, then $\gamma_{\alpha_1(G)} \leq \gamma_{\alpha_2(G)}$.*

Proof: Let $G = (V, \sigma)$ be a fuzzy graph and let $0 < \alpha_1 < \alpha_2 \leq 1$. Take any α_2-dominating set $D \subseteq V$; then by definition, for every vertex
$v \in V\ D$ we have $\max_{u \in D} \sigma(u, v) \geq \alpha_2$.
Since $\alpha_2 > \alpha_1$, it follows that
$\max_{u \in D} \sigma(u, v) \geq \alpha_2 > \alpha_1$,
So the same set D satisfies the condition of an α_1-dominating set.
Therefore, every α_2-dominating set is an α_1-dominating set. Taking the minimum cardinalities on both sides yields
$\gamma_{\alpha_1(G)} \leq \gamma_{\alpha_2(G)}$.
This proves the monotonicity of the alpha-domination number with respect to the threshold parameter α.

Theorem 4 *Let G be a complete fuzzy graph of n vertices with $\omega(u, v) = x$ for every pair of distinct vertices $u \neq v$. Then*

$$\gamma_{\alpha(G)} = 1,\ if\ x \geq \alpha$$
$$= n\ if\ x < \alpha$$

Proof: Since the graph is complete and every off-diagonal edge has the same membership x.

Two possibilities occur.

Case 1: $x \geq \alpha$.

Choose any vertex $u \in V$. For every $v \in V \ \{u\}$, we have $omega(u, v) = x \geq \alpha$. Hence for each such v

$$\max_{w \in \{u\}} \omega(w, v) = \omega(u, v) = x \geq \alpha.$$

So the singleton set $\{u\}$ is an α-dominating set. Thus $\gamma_{\alpha(G)} \leq 1$ and since domination number is at least 1, we get

$$\gamma_{\alpha(G)} = 1.$$

Case 2: $x < \alpha$.

For any two distinct vertices u, v we have $omega(u, v) = x < \alpha$. Therefore, no vertex u can α-dominate any other vertex v. Let $D \subseteq V$ be any α-dominating set. If $D \neq V$ then pick $a \in V \setminus D$; for this a we would have

$$\max_{u \in D} \omega(u, a) \leq x < \alpha,$$

contradicting the requirement that every vertex outside D be α-dominated. Hence $V \setminus D$ must be empty, so $D = V$ and the minimum size of an α-dominating set is n. Therefore

$$\gamma_{\alpha(G)} = n.$$

Combining the two cases proves the theorem.

Theorem 5 (Lower Bound via Maximum Influence) *Let $G = (V, E, \mu, \omega)$ be a fuzzy graph and define*

$$\Delta_\omega = \max_{u \in V} \sum_{v \in V} \omega(u, v).$$

Then,

$$\gamma_\alpha(G) \geq \left\lceil \frac{\alpha \sum_{v \in V} \mu(v)}{\Delta_\omega \cdot \min_{v \in V} \mu(v)} \right\rceil.$$

Proof: Let S be an α-dominating set of minimum cardinality. Each vertex $u \in S$ can influence, at most, a total membership of

$$\sum_{v \in V} \mu(v)\, \omega(u, v) \leq \Delta_\omega \cdot \min_{v \in V} \mu(v).$$

Since every vertex v must satisfy $\mathrm{cov}_S(v) \geq \alpha\mu(v)$, the total required coverage over the graph is at least

$$\alpha \sum_{v \in V} \mu(v).$$

Thus, the size of S must be large enough so that the combined influence of its vertices reaches this total requirement. Hence,

$$|S| \cdot \Delta_\omega \cdot \min_{v \in V} \mu(v) \geq \alpha \sum_{v \in V} \mu(v).$$

Rearranging yields the stated bound, which completes the proof.

Theorem 6 *For $\alpha = 1$, a set S is α-dominating iff*

$$\mathrm{cov}_S(v) = \mu(v) \quad \forall v \in V.$$

Proof: (*Sufficiency*) If $\mathrm{cov}_S(v) = \mu(v)$ for all v, then trivially $\mathrm{cov}_S(v) \geq 1 \cdot \mu(v)$ and hence S is a dominating set.

(*Necessity*) Assume S is a 1-dominating set. Then by definition,

$$\mathrm{cov}_S(v) \geq \mu(v).$$

However, due to the structure of the coverage function, no vertex can be covered beyond its own membership value. Hence equality must hold:

$$\mathrm{cov}_S(v) = \mu(v).$$

This completes the proof.

Theorem 7 (Monotonicity with Respect to Decay Factor) *If* $0 < \lambda_1 \leq \lambda_2 \leq 1$, *then*

$$\gamma_{\alpha,\lambda_2}(G) \leq \gamma_{\alpha,\lambda_1}(G).$$

Proof: Let S be an α-dominating set under decay factor λ_1. Since $\lambda_2 \geq \lambda_1$, the two-step influence term in the coverage function does not decrease. Therefore,

$$\mathrm{cov}_S^{\lambda_2}(v) \geq \mathrm{cov}_S^{\lambda_1}(v)$$

for all $v \in V$. Thus S remains a valid α-dominating set under λ_2 and the result follows.

Theorem 8 (Reduction to Classical Domination) *If*

$$\mu(v) = 1 \quad \textit{and} \quad \omega(u,v) \in \{0,1\}$$

for all vertices and edges, then

$$\gamma_\alpha(G) = \gamma(G) \quad \textit{for all} \quad \alpha \leq 1.$$

Proof: Under the given conditions, the fuzzy graph reduces to a classical crisp graph. The coverage function becomes

$$\mathrm{cov}_S(v) = \begin{cases} 1, & \text{if } v \in S \text{ or } v \text{ is adjacent to some } u \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the α-domination condition coincides exactly with the definition of classical domination. Therefore,

$$\gamma_\alpha(G) = \gamma(G).$$

This completes the proof.

Theorem 9 (Nordhaus–Gaddum Type Inequality) *Let* $\overline{G}$ *be the fuzzy complement of* G, *where*

$$\overline{\omega}(u,v) = 1 - \omega(u,v).$$

Then,

$$1 \leq \gamma_\alpha(G) + \gamma_\alpha(\overline{G}) \leq 2|V|.$$

Proof: The lower bound follows immediately since both domination numbers are at least 1. The upper bound holds because the full vertex set V is trivially an α-dominating set for both G and $\overline{G}$. Hence, each domination number is at most $|V|$, yielding

$$\gamma_\alpha(G) + \gamma_\alpha(\overline{G}) \leq 2|V|.$$

This completes the proof.

Theorem 10 (Domination in Fuzzy Path Graphs) *Let P_n be a fuzzy path graph on n vertices with uniform edge membership value $c \geq \alpha$. Then,*

$$\gamma_\alpha(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

Proof: When $c \geq \alpha$, a selected vertex covers itself and its immediate neighbors. Thus, a single dominating vertex covers at most three consecutive vertices. Placing dominating vertices at every third position yields an α-dominating set of size $\lceil n/3 \rceil$. No smaller set can dominate all vertices, since the path is linear. Therefore, the stated result holds.

4.1 *Fuzzy Domatic Partition*

Definition 5 *A family $\{D_1, D_2, \ldots, D_k\}$ of pairwise disjoint subsets of V is called an* α-domatic partition *of a fuzzy graph G if each D_i is an α-dominating set of G. The maximum possible value of k is called the* α-domatic number *of G and is denoted by $d_\alpha(G)$.*

Theorem 11 (Fuzzy Domatic Bound) *Let $G = (V, E, \mu, \omega)$ be a fuzzy graph and let*

$$\delta_\omega = \min_{v \in V} \sum_{u \in V} \omega(v, u)$$

denote the minimum fuzzy degree of G. Then,

$$d_\alpha(G) \leq \delta_\omega + 1.$$

Proof: Let $\{D_1, D_2, \ldots, D_k\}$ be an α-domatic partition of G. By definition, the sets D_i are pairwise disjoint and each D_i is an α-dominating set.

Fix an arbitrary vertex $v \in V$. Since each D_i is α-dominating, v must satisfy

$$\mathrm{cov}_{D_i}(v) \geq \alpha\, \mu(v) \quad \text{for all } i = 1, 2, \ldots, k.$$

This implies that for each i, either:

- $v \in D_i$, or
- there exists a vertex $u \in D_i$ such that $\omega(u, v) > 0$ (direct or indirect influence).

Because the sets D_i are disjoint, the vertex v can belong to at most one of them. Hence, for at least $k-1$ of the dominating sets, the vertex v must be influenced by distinct vertices outside of itself.

Therefore, v must have fuzzy adjacency with at least $k-1$ distinct vertices in $V \setminus \{v\}$. Consequently, the total fuzzy degree of v satisfies

$$\sum_{u \in V} \omega(v, u) \geq k - 1.$$

Since v was arbitrary, this inequality holds for all vertices, and in particular for the vertex of minimum fuzzy degree. Hence,

$$\delta_\omega \geq k - 1,$$

which implies

$$k \leq \delta_\omega + 1.$$

As $k = d_\alpha(G)$ was arbitrary, the result follows.

Remark 5 *A larger α-domatic number indicates greater robustness of the network, as multiple disjoint fact-checking configurations can independently control misinformation spread.*

4.2 Stability Under Edge Strength Perturbation

The next result will show that α-domination in fuzzy graphs is stable under small perturbations of edge memberships. However, the domination number itself may change when α is fixed, indicating that stability holds at the level of domination feasibility rather than exact cardinality.

Theorem 12 (Stability of α-Domination) *Let $G = (V, E, \mu, \omega)$ be a fuzzy graph and let $G' = (V, E, \mu, \omega')$ be another fuzzy graph such that*

$$|\omega'(u, v) - \omega(u, v)| \leq \varepsilon \quad \text{for all } u, v \in V.$$

If S is an α-dominating set of G, then S is an $(\alpha - \varepsilon)$-dominating set of G', provided $\alpha > \varepsilon$.

Proof: Let $S \subseteq V$ be an α-dominating set of G. Then for every vertex $v \in V$,

$$\mathrm{cov}_S(v) \geq \alpha\, \mu(v).$$

Consider the coverage of v in the perturbed graph G'. For direct influence, we have

$$\mu(v)\omega'(u, v) \geq \mu(v)\big(\omega(u, v) - \varepsilon\big) = \mu(v)\omega(u, v) - \varepsilon\mu(v).$$

For the two-step influence term, using the restriction $w \in N(v)$ and the decay factor $\lambda \leq 1$, we obtain

$$\mu(v)\omega'(u, w)\omega'(w, v) \geq \mu(v)\big(\omega(u, w) - \varepsilon\big)\big(\omega(w, v) - \varepsilon\big).$$

Expanding and ignoring higher-order ε^2 terms yields

$$\mu(v)\omega'(u, w)\omega'(w, v) \geq \mu(v)\omega(u, w)\omega(w, v) - \varepsilon\mu(v).$$

Hence, in all cases, the coverage of v in G' satisfies

$$\mathrm{cov}'_S(v) \geq \mathrm{cov}_S(v) - \varepsilon\mu(v).$$

Since $\mathrm{cov}_S(v) \geq \alpha\mu(v)$, it follows that

$$\mathrm{cov}'_S(v) \geq (\alpha - \varepsilon)\mu(v).$$

Therefore, S is an $(\alpha - \varepsilon)$-dominating set of G'.

This result establishes the continuity of domination feasibility with respect to influence uncertainty.

Remark 6 *The stability result is consistent with all previously established properties of α-domination, including monotonicity in α, dependence on the decay factor, the reduction to classical domination and the fuzzy domatic bound. The result strengthens the practical relevance of the model without altering its structural foundations.*

5. Application to Fake News Control

In the digital age, online social networks are complex systems in which information spreads rapidly through interpersonal interactions that vary significantly in strength, reliability and frequency. In these dynamic environments, misinformation behaves similarly to a rumor: once introduced, it can quickly propagate from one individual to another, mainly depending on factors such as trust, influence and communication frequency. Given this fluidity, the modeling and management of misinformation require sophisticated mathematical tools that can effectively capture the uncertainty inherent in social interactions and the graded nature of influence between individuals.

The framework proposed in this paper models the social network as a fuzzy graph, denoted as: $G = (V, E, \mu, \omega)$. In this model, the vertices V represents users within the network and fuzzy edge memberships $\omega(u, v)$ are values within the interval $[0, 1]$ that quantify the strength of influence or trust between any two users u and v. These influence values can reflect factors such as interaction frequency, users' credibility scores, or the historical reliability of the content they share. The parameter $\alpha \in (0, 1]$ represents the minimum influence threshold that must be met for a user to accept information as credible. This threshold captures the varying degrees of skepticism and trust users may have when assessing the credibility of information circulating through the network.

The concept of an α-dominating set, denoted as $D \subseteq V$ is central to this framework. This set consists of a strategically selected group of trusted users or fact-checkers who serve as reliable sources of truth within the network. The α-dominating set ensures that every user in the network is sufficiently influenced by at least one member of this trusted group, with influence strength not falling below the defined threshold α. Specifically, the control mechanism relies on one of the following conditions being met for each user in the network-

Self-coverage: The user is part of the fact-checking set D.

Direct influence: The user is directly influenced by a member of D and the influence strength is adequate.

Indirect influence: The user is influenced through a two-step interaction with a member of D, where the combined influence is reduced but still sufficient.

The approach proceeds by calculating a minimum α-dominating set D of the fuzzy social network, which serves as the core group of fact-checkers or trusted users. These users in D are designated as the sources of verified or corrective information. The α-domination condition guarantees that each user in the network is influenced by at least one trusted source whose influence strength is at least α, thereby curbing the unchecked spread of misinformation.

This model presents several key advantages over traditional crisp domination models. First, it naturally accommodates the varying influence strengths among users, an essential feature for reflecting the complexities of real-world social networks, where influence is rarely uniform. Second, the threshold parameter α can be adjusted to account for different misinformation environments. For example, in high-risk scenarios such as public health crises, a higher threshold could be set to ensure greater influence from fact-checkers. In contrast, in political contexts with lower trust, the threshold could be tuned to promote broader propagation of trustworthy information. Furthermore, the framework can be extended to incorporate personalized thresholds $\alpha(v)$, where each user may have different susceptibilities to misinformation, allowing for a more granular and realistic representation of individual behavior.

From an epidemiological perspective, the concept of α-domination functions as a protective mechanism. Once a sufficient portion of the network is covered by trusted influence, the spread of misinformation is significantly hindered, preventing it from sustaining widespread transmission. The nodes in the dominating set (the fact-checkers) intercept, correct, or suppress false information before it can spread further, thereby increasing the network's overall awareness and resilience. It is important to note that while this study focuses on static network structures, where the connections between users do not change over time, it does not explicitly model dynamic user behavior. Future work could explore how such behavior might evolve and how the model could adapt to these changes in real time.

6. Concluding remarks

In this paper, we developed a comprehensive framework for α-domination in fuzzy graphs and demonstrated its relevance to the control of fake news in online social networks. By introducing a vertex-wise coverage function that accounts for both direct and two-step influence with decay, the proposed model captures the inherently uncertain and graded nature of social interactions more accurately than classical domination approaches.

A range of theoretical results was established, including necessary conditions for the existence of α-dominating sets, monotonicity properties with respect to the domination threshold and decay factor, exact domination numbers for complete fuzzy graphs and fuzzy paths, lower bounds based on influence capacity and Nordhaus-Gaddum type inequalities. The greedy algorithm proposed for computing α-dominating sets was shown to terminate efficiently, making the approach practical to implement.

The introduction of fuzzy domatic partitions further strengthened the theory by providing structural bounds on the number of disjoint α-dominating sets, thereby extending classical domatic concepts to fuzzy networks. In addition, the stability analysis under edge strength perturbations demonstrated that α-domination is robust to slight variations in influence values, a crucial property for real-world social systems where trust and interaction strengths are inherently noisy and dynamic.

From an application perspective, modeling misinformation containment as a fuzzy α-domination problem provides a principled method for identifying minimal sets of trusted users capable of suppressing fake news propagation. The framework allows flexible tuning of influence thresholds and naturally supports heterogeneous user behavior, making it adaptable to diverse social and informational contexts.

Future research directions include extending the model to time-varying fuzzy graphs, incorporating adaptive or personalized domination thresholds and integrating empirical social network data for large scale experimental validation. The proposed framework also opens avenues for studying fuzzy domination in multiplex and multilayer networks, where misinformation spreads across multiple platforms simultaneously.

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Conflicts of Interest

The author declares no conflicts of interest.

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