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Archimedean T-norm Based q-Rung Orthopair Fuzzy Hamy Mean Operator With Ordinal Priority Weights

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ABSTRACT

Archimedean t-norm provides an advantage to represent the formal structure of aggregation operators (AOs) in a simpler form i.e., from a multivariate to univariate function through an additive generator. This study focuses on q-rung orthopair fuzzy numbers, a field typically governed by t-norms. So in this paper, we introduced Hamy mean and weighted Hamy mean operators for q-rung orthopair fuzzy numbers under a generalized construction of continuous and Archimedean classes of t-norms. To further develop a comprehensive AO-based multi-criteria group decision-making model, an objective weight determination approach, the q-rung orthopair fuzzy Archimedean ordinal priority approach, is used to systematically evaluate attribute weights. Finally, the constructed approach is demonstrated in practice through a detailed illustration of a real-world decision-making problem.

1. Introduction

The integration of fuzzy sets into multi-criteria group decision making (MCGDM) represents a significant advancement, effectively addressing the inherent uncertainty often encountered in many decision-making situations. Fuzzy set [34] establishes a mathematical framework for dealing with imprecision and ambiguity, making it particularly well-suited for applications in decision science. An extension of fuzzy sets, the intuitionistic fuzzy set (IFS) [2] emerged as a more effective means of handling uncertainty. It encompasses membership and non-membership degrees along with the hesitation degree, offering decision makers (DMs) a more refined depiction of uncertainty. Building upon these advancements, Pythagorean fuzzy set (PFS) [32] and then Fermentean fuzzy set (FFS) [27], offer

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a more comprehensive framework. This enhancement enables DMs to articulate their assessments clearly, leading to a precise depiction of the contexts. In recent years, q-rung orthopair fuzzy set (q-ROFS) [33] has emerged as a versatile extension of the orthopair family of fuzzy sets. It introduces a parameter q to regulate the degree of hesitancy, providing DMs with a flexible means of handling vagueness in decision-making. This generalized framework enables DMs to capture and reflect the involved vagueness in human perceptions and the linguistic representation used for attribute evaluation.

Archimedean t-norms (ATNs) (or Archimedean t-conorms (ATCNs)) provide a flexible mathematical construction for modelling the *and* and *or* notions in the fuzzy set framework. ATNs are not necessarily continuous; a continuous ATN is strictly increasing on the subset where its value is greater than zero [4]. One key advantage of continuous ATNs and ATCNs is their representation via an additive generator (AG), which simplifies computation by transforming a multivariate function into its univariate generator [12]. Over time, researchers have used these ATNs and ATCNs to develop aggregation operators (AOs) for decision-making methods [3, 29, 30]. AOs are essential for integrating various sources of information and for representing preferences for attributes in decision-making. Specifically, weighted AOs have been instrumental in aggregating conflicting criteria and preferences within fuzzy MCGDM contexts [3]. In this row, the WA operator for q-run orthopair fuzzy numbers (q-ROFNs) has been extensively studied by various researchers under different t-norms that is Liu and Wang [14] used algebraic t-norm, Jana *et al.* [11] applied Dombi t-norm, Darko and Liang [6] utilized Hamacher t-norm, Seikh and Mandal [24] used Frank t-norm, and Senapati *et al.* [26] implemented Aczél-Alsina t-norm, all these t-norms are continuous Archimedean. Similarly, various other AOs for q-ROFNs are also studied under such continuous ATNs [16].

On the other side, Liu and Wang [13] define some ATN-based arithmetic operations for q-ROFNs and utilize them to develop Bonferroni mean (BM) and weighted BM operators. Such a construction provides a general architecture in an AO to aggregate a finite number of q-ROFNs. In this regard, Qin *et al.* [19] constructed the Archimedean power partitioned Muirhead mean (MM) and weighted MM operators of q-ROFNs. In a similar manner, Qin *et al.* [20] proposed the q-ROF Archimedean power partitioned weighted BM operator and developed an MCDM approach based on the suggested AO. Further, to decrease the deviation caused by the subjective perspective of the DM in the MCGDM problems, Shao *et al.* [28] introduced the confidence q-ROF Archimedean weighted averaging (WA), weighted geometric (WG), ordered weighted averaging (OWA), and ordered weighted geometric (OWG) operators. Ai *et al.* [1] suggested a new representation of Archimedean arithmetic operations of q-ROFNs from the perspective of q-rung orthopair fuzzy (q-ROF) representation theorem, and the same representation is used in this chapter for avoiding failure of $\psi(t) = \phi(1-t)$ condition (see subsection 2.3). Recently, Seikh and Mandal [25] utilized Ai *et al.*'s [1] framework of Archimedean arithmetic operations for developing the WA operator under the q-rung orthopair fuzzy environment (q-ROFE). It is evident from the literature that there is no construction of the Hamy mean (HM) operator within the general framework of ATN and ATCN for any orthopair fuzzy information. The Hammy symmetric function, now known as HM, can handle correlations among k-attributes via a parameter k. Hara *et al.* [10] discussed a refinement of the arithmetic mean (AM) and geometric mean (GM) through an inequality and showed the existence of HM between AM and GM. Therefore, HM is also considered a generalization of AM and GM, making it an important player in the domain of AO. Thus, we developed the HM and its weighted variant for q-ROFNs under ATN and ATCN setups.

A crucial aspect of weighted AOs is the assignment of weights to attributes, which significantly impacts the decision-making process. Over time, researchers have explored various methods to determine appropriate weights for different attributes based on their relative significance [5]. To evaluate the significance of each attribute, various objective weight-finding techniques are helpful for DMs. One such method is the ordinal priority approach (OPA), which utilizes a linear programming model to compute attributes' weights. Deveci *et al.* [8] introduced the q-ROF-OPA, while Pamucar *et al.* [15] proposed an OPA based on Schweizer-Sklar norms. Recently, Rawat and Komal [21] suggested a q-ROF-OPA based on ATNs and ATCNs, i.e., q-ROF Archimedean OPA (q-ROFAOPA). In this paper, we developed an optimization model based on OPA under ATNs and ATCNs for q-ROFNs, incorporating the proposed q-ROF Archimedean weighted Hamy mean (q-ROFAWHM) operator to introduce an MCGDM method. Finally, we discussed its real-world application to the selection of rehabilitation strategies using the q-ROFAWHM operator and the q-ROF Archimedean OPA (q-ROFAOPA)-based MCGDM model. The suggested technique addresses complex decision-making scenarios in which multiple DMs and criteria jointly determine the final outcome. This MCGDM framework offers a structured approach to facilitating group decision-making by incorporating the assessments and competence of several DMs. Since DMs often face challenges in evaluating and ordering alternatives across multiple criteria, the primary goal of the proposed MCGDM is to ensure a well-informed, comprehensive decision that reflects the expert group's varying perspectives and priorities. This approach not only integrates individual preferences using the q-ROFAWHM aggregation operator but also addresses conflicts and inherent uncertainties in the decision process [17, 23, 31]. Additionally, the methodology employs an AO named q-ROFAWHM and an optimization technique named q-ROFAOPA to identify a satisfactory criterion for importance.

The paper is structured as follows: In section 2, we discuss preliminary concepts, including q-ROFS, ATN, ATCN, HM, and Archimedean operations of q-ROFNs. In section 3, we introduced the q-ROF Archimedean HM (q-ROFAHM) and q-ROF Archimedean weighted HM (q-ROFAWHM) operators, along with some specific cases of the q-ROFAWHM operator. In section 4, the objective weights evaluation technique, i.e., q-ROFAOPA, is discussed. In section 5, we proposed an MCGDM method based on the q-ROFAOPA and q-ROFAWHM operators; additionally, its applicability is also illustrated by analyzing a practical problem of strategy selection related to rehabilitation. Finally, section 6 discussed some concluding remarks on the paper.

2. Preliminary

2.1 q-Rung Orthopair Fuzzy Set

Definition 1 ([33]) A q-ROFS A on a domain of discourse $\mathbb{D}$ is a collection of elements with their orthopair membership grades $(\mu_A(x), \nu_A(x))$, which is defined as

$$A = \{(x, (\mu_A(x), \nu_A(x))) \mid x \in \mathbb{D}\} \tag{1}$$

where $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ indicates support for and support against membership of x in A and satisfies the q^{th} degree inequality $\mu_A^q(x) + \nu_A^q(x) \leq 1, q \geq 1$.

The $\pi_A(x) = \sqrt[q]{1 - (\mu_A(x))^q - (\nu_A(x))^q}$ is the hesitancy of x in A. The $(\mu_A, \nu_A) \in \mathcal{Q}$ is called q-ROFN, where $\mathcal{Q}$ denotes the set of q-ROFNs.

Note 1: For $q =$ 1, 2, and 3, the q-ROFS is reduced to the IFS [2], PFS [32], and FFS [27], respectively, and their geometric representation can be seen through Figure 1.

2.2 Score and Accuracy Functions

Definition 2 ([14]) Suppose $\alpha_i = (\mu_{\alpha_i}, \nu_{\alpha_i})$ or simply (μ_i, ν_i) is a q-ROFN; the score function S for q-ROFNs is a real-valued function that is defined as follows:

$$S(\alpha_i) = \mu_i^q - \nu_i^q \tag{2}$$

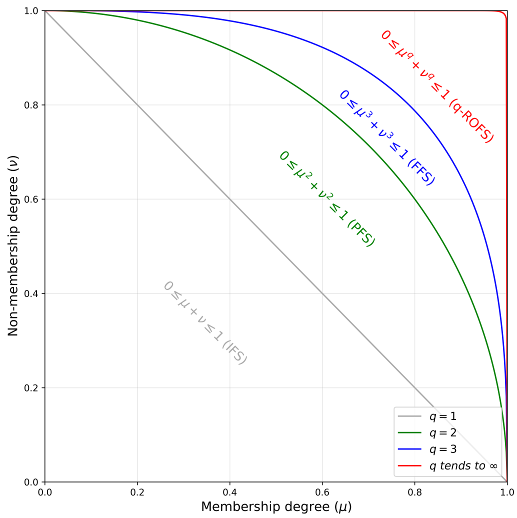

Figure 1: Geometrical representation of q-rung orthopair fuzzy values

where $S(\alpha_i) \in [-1, 1]$ and higher the score value $S(A)$ means bigger the q-ROFN.

Definition 3 ([14]) Suppose $\alpha_i = (\mu_i, \nu_i)$ is a q-ROFN; the accuracy function H for q-ROFNs is a real-valued function that is defined as follows:

$$H(\alpha_i) = \mu_i^q + \nu_i^q \tag{3}$$

where $H(\alpha_i) \in [0, 1]$ a greater accuracy value ensures a larger q-ROFN. Usually, the accuracy of q-ROFNs is calculated only when their score values are the same. Further, using the score and accuracy functions, an order can be defined for q-ROFNs.

Let α_1 and α_2 be q-ROFNs. Then the order between these two values is defined as follows [14]:

1. If $S(\alpha_1) < S(\alpha_2) \implies \alpha_1 < \alpha_2$.
2. If $S(\alpha_1) = S(\alpha_2)$, then compare their accuracy values
   (a) If $H(\alpha_1) < H(\alpha_2) \implies \alpha_1 < \alpha_2$.
   (b) If $H(\alpha_1) = H(\alpha_2) \implies \alpha_1 = \alpha_2$.

Several studies report score functions that can directly rank q-ROFNs, thereby eliminating the need for a separate accuracy function. Adding to this discourse, Rawat *et al.* [22] have introduced a new score function, articulated as follows:

$$Sc(\alpha_i) = \frac{1}{2}\left((\mu_i^q - \nu_i^q + 1) - \frac{1}{\pi^3} \cos\left((1 - \pi_i^q) \frac{\pi}{2} \right) \right) \tag{4}$$

where π_i is a hesitancy associated with a q-ROFN.

2.3 *Archimedean T-norm and T-conorm*

Definition 4 ([4]) A t-norm T is said to be an ATN if for every $(x, y) \in (0, 1)^2$ $\exists$ a $k \in \mathbb{N}$ s.t. $T(\overbrace{x, \ldots, x}^{k-times}) < y$. That is, for any $a \in (0, 1)$ $\lim_{k \to \infty} T_k(\overbrace{a, \ldots, a}^{k-times}) = 0$ and, only 0 and 1 are the idempotent elements of T.

ATNs need not be continuous. Moreover, the existence of a limit alone is not sufficient to ensure the continuity of a t-norm.

A continuous ATN T can be defined by its strictly decreasing AG $\phi : [0,1] \to [0,\infty]$ s.t. $\phi(1) = 0$, as follows:

$$T(x,y) = \phi^{-1}(\phi(x) + \phi(y)) \tag{5}$$

Definition 5 ([4]) A t-norm S is said to be an ATCN if for every $(x,y) \in (0,1)^2$ $\exists$ a $k \in \mathbb{N}$ with $S(\overbrace{x,\ldots,x}^{k-times}) > y$.

A continuous ATCN S can be defined by its strictly increasing AG $\psi : [0,1] \to [0,\infty]$ with $\psi(0) = 0$, as follows:

$$S(x,y) = \psi^{-1}(\psi(x) + \psi(y)) \tag{6}$$

For an AG $\phi : [0,1] \to [0,\infty]$ of a t-norm. The AG of t-conorm S, $\psi : [0,1] \to [0,\infty]$, is given by $\psi(t) = \phi(1-t)$.

In fuzzy set theory, t-norms serve as the intersection operation, and t-conorms as the union operation. Various t-norms and t-conorms, such as Aczél-Alsina, Dombi, Frank, and Hamacher, are used to define basic operations for q-ROFNs. Table 1 present some continuous q-ROF ATNs given in [1] from the perspectives of q-ROF representation theorem. That are, algebraic (T^A), Aczél-Alsina $(T^{AA})_{\gamma\in(0,\infty)}$, Dombi $(T^D)_{\gamma\in(0,\infty)}$, Frank $(T^F)_{\gamma\in(0,\infty]}$, and Hamacher $(T^H)_{\gamma\in[0,\infty)}$ t-norms with their continuous and strictly decreasing AGs [12].

Table 1: Archimedean t-norms for q-ROFNs

Name	Triangular norm (T)	Additive generator (ϕ)
Algebraic	$T^A(x^q,y^q) = x \cdot y$	$\phi^A(t^q) = -\log t^q$
Aczél-Alsina	$T^{AA}_{\gamma\in(0,\infty)}(x^q,y^q) = \left(e^{-((-\log x^q)^\gamma + (-\log y^q)^\gamma)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}$	$\phi^{AA}_{\gamma\in(0,\infty)}(t^q) = (-\log t^q)^\gamma$
Dombi	$T^D_{\gamma\in(0,\infty)}(x^q,y^q) = \left(\frac{1}{1+\left(\left(\frac{1-x^q}{x^q}\right)^\gamma + \left(\frac{1-y^q}{y^q}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}$	$\phi^D_{\gamma\in(0,\infty)}(t^q) = \left(\frac{1-t^q}{t^q}\right)^\gamma$
Frank	$T^F_{\gamma\in(0,\infty]}(x^q,y^q) = \begin{cases} x \cdot y, & \gamma = 1; \\ \max\left((x^q+y^q-1)^{\frac{1}{q}}, 0\right), & \gamma = \infty; \\ \left(\log_\gamma\left(1 + \frac{(\gamma^{x^q}-1)(\gamma^{y^q}-1)}{(\gamma-1)}\right)\right)^{\frac{1}{q}}, & \text{else.} \end{cases}$	$\phi^F_{\gamma\in(0,\infty]}(t^q) = \begin{cases} -\log t^q, & \gamma = 1; \\ 1-t^q, & \gamma = \infty; \\ \log\left(\frac{\gamma-1}{\gamma^{t^q}-1}\right), & \text{else.} \end{cases}$
Hamacher	$T^H_{\gamma\in[0,\infty)}(x^q,y^q) = \begin{cases} 0, & \gamma = u = v = 0; \\ \left(\frac{x^q y^q}{\gamma + (1-\gamma)(x^q+y^q-x^q y^q)}\right)^{\frac{1}{q}}, & \text{else.} \end{cases}$	$\phi^H_{\gamma\in[0,\infty)}(t^q) = \begin{cases} \frac{1-t^q}{t^q}, & \gamma = 0; \\ \log\left(\frac{\gamma+(1-\gamma)t^q}{t^q}\right), & \text{else.} \end{cases}$

Similarly, Table 2 contains some continuous q-ROF ATCNs (S^A, $S^{AA}_{\gamma\in(0,\infty)}$, $S^D_{\gamma\in(0,\infty)}$, $S^F_{\gamma\in(0,\infty]}$, and $S^H_{\gamma\in[0,\infty)}$) with continuous and strictly increasing AGs from the perspectives of the q-ROF representation theorem [1].

Table 2: Archimedean t-conorms for q-ROFNs

Name	Triangular cornorm (S)	Additive generator (ψ)
Algebraic	$S^A(x^q, y^q) = (x^q + y^q - x^q y^q)^{\frac{1}{q}}$	$\psi^A(t^q) = -\log(1 - t^q)$
Aczél-Alsina	$S^{AA}_{\gamma\in(0,\infty)}(x^q, y^q) = \left(1 - e^{-((-\log(1-x^q))^{\gamma} + (-\log(1-y^q))^{\gamma})^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}$	$\psi^{AA}_{\gamma\in(0,\infty)}(t^q) = (-\log(1 - t^q))^{\gamma}$
Dombi	$S^{D}_{\gamma\in(0,\infty)}(x^q, y^q) = \left(1 - \dfrac{1}{1 + \left(\left(\frac{x^q}{1-x^q}\right)^{\gamma} + \left(\frac{y^q}{1-y^q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}$	$\psi^{D}_{\gamma\in(0,\infty)}(t^q) = \left(\frac{t^q}{1-t^q}\right)^{\gamma}$
Frank	$S^{F}_{\gamma\in(0,\infty]}(x^q, y^q) = \begin{cases} (x^q + y^q - x^q y^q)^{\frac{1}{q}}, & \gamma = 1; \\ \min\left((x^q + y^q)^{\frac{1}{q}}, 1\right), & \gamma = \infty; \\ \left(1 - \log_{\gamma}\left(1 + \frac{(\gamma^{1-x^q} - 1)(\gamma^{1-y^q} - 1)}{(\gamma - 1)}\right)\right)^{\frac{1}{q}}, & \text{else.} \end{cases}$	$\psi^{F}_{\gamma\in(0,\infty]}(t^q) = \begin{cases} -\log(1 - t^q), & \gamma = 1; \\ t^q, & \gamma = \infty; \\ \log\left(\frac{\gamma - 1}{\gamma^{1-t^q} - 1}\right), & \text{else.} \end{cases}$
Hamacher	$S^{H}_{\gamma\in[0,\infty)}(x^q, y^q) = \begin{cases} 1, & \gamma = 0 \text{ and } u = v = 1; \\ \left(\frac{x^q + y^q - x^q y^q - (1-\gamma)x^q y^q}{1 - (1-\gamma)x^q y^q}\right)^{\frac{1}{q}}, & \text{else.} \end{cases}$	$\psi^{H}_{\gamma\in[0,\infty)}(t^q) = \begin{cases} \frac{t^q}{1-t^q}, & \gamma = 0; \\ \log\left(\frac{\gamma + (1-\gamma)(1-t^q)}{1-t^q}\right), & \text{else.} \end{cases}$

2.4 *Archimedean T-norm and T-conorm Based Operations*

Let α_1 and α_2 be two q-ROFNs, and let ϕ and ψ are strictly decreasing and increasing AG of T and S, respectively, and $\lambda > 0$ [1].

1. $\alpha_1 \oplus \alpha_2 = \left((\psi^{-1}(\psi(\mu_1^q) + \psi(\mu_2^q)))^{\frac{1}{q}}, (\phi^{-1}(\phi(\nu_1^q) + \phi(\nu_{\alpha_2}^q)))^{\frac{1}{q}}\right)$;
2. $\alpha_1 \otimes \alpha_2 = \left((\phi^{-1}(\phi(\mu_1^q) + \phi(\mu_2^q)))^{\frac{1}{q}}, (\psi^{-1}(\psi(\nu_1^q) + \psi(\nu_2^q)))^{\frac{1}{q}}\right)$;
3. $\lambda\alpha_1 = \left((\psi^{-1}(\lambda\psi(\mu_1^q)))^{\frac{1}{q}}, (\phi^{-1}(\lambda\phi(\nu_1^q)))^{\frac{1}{q}}\right)$;
4. $\alpha_1^{\lambda} = \left((\phi^{-1}(\lambda\phi(\mu_1^q)))^{\frac{1}{q}}, (\psi^{-1}(\lambda\psi(\nu_1^q)))^{\frac{1}{q}}\right)$.

The results obtained by the above-mentioned rules are q-ROFNs. Moreover, the first two operations can be generalized to a finite collection of q-ROFNs.

1. $\bigoplus_{i=1}^{n} \alpha_i = \left(\left(\psi^{-1}\left(\sum_{i=1}^{n} \psi(\mu_i^q)\right)\right)^{\frac{1}{q}}, \left(\phi^{-1}\left(\sum_{i=1}^{n} \phi(\nu_i^q)\right)\right)^{\frac{1}{q}}\right)$;
2. $\bigotimes_{i=1}^{n} \alpha_i = \left(\left(\phi^{-1}\left(\sum_{i=1}^{n} \phi(\mu_i^q)\right)\right)^{\frac{1}{q}}, \left(\psi^{-1}\left(\sum_{i=1}^{n} \psi(\nu_i^q)\right)\right)^{\frac{1}{q}}\right)$.

Let α_1, and α_2 are two q-ROFNs and let λ_1, $\lambda_2 > 0$, then some fundamental properties hold by these operations are as follows [13]:

(i) $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;

(ii) $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;

(iii) $\lambda_1(\alpha_1 \oplus \alpha_2) = \lambda_1\alpha_1 \oplus \lambda_1\alpha_2$;

(iv) $\alpha_1^{\lambda_1} \otimes \alpha_2^{\lambda_1} = (\alpha_1 \otimes \alpha_2)^{\lambda_1}$;

(v) $\lambda_1\alpha_1 \oplus \lambda_2\alpha_1 = (\lambda_1 + \lambda_2)\alpha_1$;

(vi) $\alpha_1^{\lambda_1} \otimes \alpha_1^{\lambda_2} = \alpha_1^{\lambda_1+\lambda_2}$.

By choosing different ATNs (and their associated t-conorms) together with the corresponding AGs ϕ and ψ shown in Table 1 and Table 2, different operational rules for q-ROFNs can be generated.

2.5 Hamy Mean

Definition 6 ([9]) Given any set $A = \{a_1, a_2, \ldots, a_n\} \subset \mathbb{R}^+$ and a granularity parameter $k \in \mathbb{N}$ and $k \leq n$, the HM on A is defined as follows:

$$\text{HM}^k(a_1, a_2, \ldots, a_n) = \frac{1}{C_n^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \left(\prod_{j=1}^{k} a_{i_j} \right)^{\frac{1}{k}} \tag{7}$$

where, $i_1, i_2, \ldots, i_k \in \mathbb{N}$ such that $1 \leq i_1 < \ldots < i_k \leq n$ and $C_n^k = \dfrac{n!}{k!(n-k)!}$.

Two special cases of the HM operator corresponding to two different values of the parameter k are shown below.

1. For $k = 1$, then the HM will convert into the AM:

$$\text{HM}^1(a_1, a_2, \ldots, a_n) = \frac{1}{n} \sum_{i=1}^{n} a_i.$$

2. For $k = n$, then the HM will convert into the GM:

$$\text{HM}^n(a_1, a_2, \ldots, a_n) = \left(\prod_{i=1}^{n} a_i \right)^{\frac{1}{n}}.$$

3. Archimedean T-norm Based Hamy Mean for q-ROFNs

3.1 q-Rung Orthopair Fuzzy Archimedean Hamy Mean

Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a collection of q-ROFNs. Then, the q-ROFAHM operator on a given set is a map from $\mathcal{Q}^n$ to $\mathcal{Q}$ and is defined as follows:

$$\text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{C_n^k} \left(\bigoplus_{1 \leq i_1 < \ldots < i_k \leq n} \left(\bigotimes_{j=1}^{k} \alpha_{i_j} \right)^{\frac{1}{k}} \right) \tag{8}$$

where $k \in \mathbb{N}_n$ is a parameter of granularity and $C_n^k = \dfrac{n!}{k!\,(n-k)!}$.

Theorem 1 *For a finite collection of q-ROFNs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, the aggregated value by the q-ROFAHM operator is a q-ROFN and is expressed as follows:*

$$\text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\left(\psi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \psi \left(\phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}}, \right.$$
$$\left. \left(\phi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \phi \left(\psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right) \tag{9}$$

Proof. We have

$$\alpha_{i_1} \otimes \alpha_{i_2} = \left(\left(\phi^{-1} \left(\phi \left(\mu_{i_1}^q \right) + \phi \left(\mu_{i_2}^q \right) \right) \right)^{\frac{1}{q}}, \left(\psi^{-1} \left(\psi \left(\nu_{i_1}^q \right) + \psi \left(\nu_{i_2}^q \right) \right) \right)^{\frac{1}{q}} \right)$$

and using mathematical induction, we get

$$\bigotimes_{j=1}^{k} \alpha_{i_j} = \left(\left(\phi^{-1} \left(\sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right)^{\frac{1}{q}}, \left(\psi^{-1} \left(\sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right)^{\frac{1}{q}} \right)$$

Then,

$$\left(\bigotimes_{j=1}^{k} \alpha_{i_j} \right)^{\frac{1}{k}} = \left(\left(\phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right)^{\frac{1}{q}}, \left(\psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right)^{\frac{1}{q}} \right)$$

Further,

$$\bigoplus_{1 \le i_1 < \ldots < i_k \le n} \left(\bigotimes_{j=1}^{k} \alpha_{i_j} \right)^{\frac{1}{k}} = \left(\left(\psi^{-1} \left(\sum_{1 \le i_1 < \ldots < i_k \le n} \psi \left(\phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}}, \right.$$
$$\left. \left(\phi^{-1} \left(\sum_{1 \le i_1 < \ldots < i_k \le n} \phi \left(\psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right)$$

Finally,

$$\text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{C_n^k} \left(\bigoplus_{1 \le i_1 < \ldots < i_k \le n} \left(\bigotimes_{j=1}^{k} \alpha_{i_j} \right)^{\frac{1}{k}} \right)$$

$$= \left(\left(\psi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n} \psi \left(\phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}}, \right.$$
$$\left. \left(\phi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n} \phi \left(\psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \right). \tag{10}$$

Since ϕ and ψ are continuous strictly decreasing and increasing functions from $[0, 1]$ to $[0, \infty]$, respectively, such that $\psi(t) = \phi(1 - t)$ which implies $\phi^{-1}(t) = 1 - \psi^{-1}(t)$. This shows that

$$0 \le \left(\psi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n} \psi \left(\phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}},$$
$$\left(\phi^{-1} \left(\frac{1}{C_n^k} \sum_{1 \le i_1 < \ldots < i_k \le n} \phi \left(\psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) \right) \right) \right)^{\frac{1}{q}} \le 1$$

and we know $\mu_{i_j}^q \le 1 - \nu_{i_j}^q \Rightarrow \phi(\mu_{i_j}^q) \ge \phi(1 - \nu_{i_j}^q) = \psi(\nu_{i_j}^q)$

$$\Rightarrow \phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \phi \left(\mu_{i_j}^q \right) \right) \le \phi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right) = 1 - \psi^{-1} \left(\frac{1}{k} \sum_{j=1}^{k} \psi \left(\nu_{i_j}^q \right) \right)$$

$$\Rightarrow \psi\left(\phi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\phi\left(\mu_{i_j}^{q}\right)\right)\right) \leq \psi\left(1-\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\nu_{i_j}^{q}\right)\right)\right)$$

$$= \phi\left(1-\left(1-\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\nu_{i_j}^{q}\right)\right)\right)\right)$$

$$\Rightarrow \psi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\psi\left(\phi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\phi\left(\mu_{i_j}^{q}\right)\right)\right)\right)$$

$$\leq \psi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\phi\left(\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\nu_{i_j}^{q}\right)\right)\right)\right)$$

$$= 1-\phi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\phi\left(\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\nu_{i_j}^{q}\right)\right)\right)\right)$$

$$\Rightarrow 0 \leq \left(\psi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\psi\left(\phi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\phi\left(\mu_{i_j}^{q}\right)\right)\right)\right)\right)^{\frac{1}{q}\cdot q} +$$

$$\left(\phi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\phi\left(\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\nu_{i_j}^{q}\right)\right)\right)\right)\right)^{\frac{1}{q}\cdot q} \leq 1$$

Hence, the inequality in the q-ROFS condition is satisfied. That is, the resultant value from the q-ROFAHM operator will always be a q-ROFN.

Further, properties such as monotonicity, idempotency, and boundedness are satisfied by the q-ROFAHM operator and are discussed hereafter.

1. Boundedness In a q-ROFNs set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, suppose $\alpha^- = \left(\min_i \mu_i, \max_i \nu_i\right)$ and $\alpha^+ = \left(\max_i \mu_i, \min_i \nu_i\right)$, $i \in \mathbb{N}_n$. The resultant value of q-ROFAHM operator satisfy the following,

$$\alpha^- \leq \text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \alpha^+.$$

2. Idempotency Given a set of q-ROFNs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ s.t. $\alpha_i = \alpha = (\mu, \nu),\ \forall\ i \in \mathbb{N}_n$, the proposed AO holds the following,

$$\text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \alpha.$$

3. Monotonicity Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\alpha_1', \alpha_2', \ldots, \alpha_n'\}$ are two different collections of q-ROFNs with each $\alpha_i \leq \alpha_i'$ i.e., $\mu_i \leq \mu_i'$ and $\nu_i \geq \nu_i'$. Then the aggregated values will preserve this order i.e.,

$$\text{q-ROFAHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) \leq \text{q-ROFAHM}^k(\alpha_1', \alpha_2', \ldots, \alpha_n').$$

3.2 q-Rung Orthopair Fuzzy Archimedean Weighted Hamy Mean

Suppose $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a set of q-ROFNs with an associated weight vector $w = (w_1, w_2, \ldots, w_n) \in [0,1]^n$ s.t. $\sum_{i=1}^{n} w_i = 1$. Then the q-ROFAWHM operator on a given set with weighting vector w is a map from $\mathcal{Q}^n$ to $\mathcal{Q}$ and is given by

$$\text{q-ROFAWHM}^k(\alpha_1, \alpha_2, \ldots, \alpha_n) = \frac{1}{C_n^k}\left(\bigoplus_{1\leq i_1<\ldots<i_k\leq n}\left(\bigotimes_{j=1}^{k} w_{i_j}\alpha_{i_j}\right)^{\frac{1}{k}}\right) \quad (11)$$

Theorem 2 *For a finite collection of q-ROFNs $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, the aggregated value by the q-ROFAWHM operator is a q-ROFN and the q-ROFAWHM operator is expressed as follows:*

q-ROFAWHM$^k(\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$= \left(\left(\psi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\psi\left(\phi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\phi\left(\psi^{-1}\left(w_{i_j}\psi\left(\mu_{i_j}^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}},\right.$$
$$\left.\left(\phi^{-1}\left(\frac{1}{C_n^k}\sum_{1\leq i_1<\ldots<i_k\leq n}\phi\left(\psi^{-1}\left(\frac{1}{k}\sum_{j=1}^{k}\psi\left(\phi^{-1}\left(w_{i_j}\phi\left(\nu_{i_j}^q\right)\right)\right)\right)\right)\right)\right)^{\frac{1}{q}}\right) \quad (12)$$

All three properties of the q-ROFAHM operator are also satisfied by the q-ROFAWHM operator. In the context of the conjunction and disjunction modelling of the aggregated q-ROFNs, Equation 12 represents a general form of weighted HM operator. Thus any specific pair of AGs of continuous ATN and ATCN (Table 1 and Table 2) will give rise to a particular t-norm-based weighted HM operator as shown below.

1. Algebraic: For $\phi(t) = -\log t$ and $\psi(t) = -\log(1-t)$, the q-ROFAWHM operator will change into the q-ROF weighted HM operator.

$$\mathrm{q-ROFAWHM}_{TA}^k(\alpha_1, \alpha_2, \ldots, \alpha_n)$$
$$= \left(\left(1 - \prod_{1\leq i_1<\ldots<i_k\leq n}\left(1 - \prod_{j=1}^{k}\left(1-\left(1-\mu_{i_j}^q\right)^{w_{i_j}}\right)^{\frac{1}{k}}\right)^{\frac{1}{C_n^k}}\right)^{\frac{1}{q}},\right.$$
$$\left.\left(\prod_{1\leq i_1<\ldots<i_k\leq n}\left(1-\prod_{j=1}^{k}\left(1-(\nu_{i_j}^q)^{w_{i_j}}\right)^{\frac{1}{k}}\right)\right)^{\frac{q}{C_n^k}}\right) \quad (13)$$

2. Aczél-Alsina: For $\phi(t) = (-\log t)^\gamma$ and $\psi(t) = (-\log(1-t))^\gamma$, the q-ROFAWHM operator will reduce to the q-ROF Aczél-Alsina weighted HM operator.

$$
\mathrm{q}-\mathrm{ROFAWHM}^{k}_{T^{AA}}(\alpha_1,\alpha_2,\ldots,\alpha_n)
$$

$$
=\left(\left(1-e^{-\left(\frac{1}{C_n^k}\left(\sum_{1\le i_1<\ldots<i_k\le n}\left(-\log\left(1-e^{-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\left(-\log\left(1-e^{-\left(w_{i_j}\left(-\log(1-\mu_{i_j}^{q})\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}},\right.
$$

$$
\left.\left(e^{-\left(\frac{1}{C_n^k}\left(\sum_{1\le i_1<\ldots<i_k\le n}\left(-\log\left(1-e^{-\left(\frac{1}{k}\left(\sum_{j=1}^{k}\left(-\log\left(1-e^{-\left(w_{i_j}\left(-\log\nu_{i_j}^{q}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}}\right)\right)^{\gamma}\right)\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\right) \tag{14}
$$

3. Dombi: For $\phi(t)=\left(\frac{1-t}{t}\right)^{\gamma}$ and $\psi(t)=\left(\frac{t}{1-t}\right)^{\gamma}$, the q-ROFAWHM operator will reduce to the q-ROF Dombi weighted HM operator.

$$
\mathrm{q}-\mathrm{ROFAWHM}^{k}_{T^{D}}(\alpha_1,\alpha_2,\ldots,\alpha_n)
$$

$$
=\left(\left(1-\left(1+\left(\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(w_{i_j}\left(\frac{\mu_{i_j}^{q}}{1-\mu_{i_j}^{q}}\right)^{\gamma}\right)^{-1}\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}\right)^{\frac{1}{q}},\right.
$$

$$
\left.\left(\left(1+\left(\frac{1}{C_n^k}\sum_{1\le i_1<\ldots<i_k\le n}\left(\frac{1}{k}\sum_{j=1}^{k}\left(w_{i_j}\left(\frac{1-\nu_{i_j}^{q}}{\nu_{i_j}^{q}}\right)^{\gamma}\right)^{-1}\right)^{-1}\right)^{\frac{1}{\gamma}}\right)^{-1}\right)^{\frac{1}{q}}\right) \tag{15}
$$

4. Frank: For $\phi(t)=\log\left(\frac{\gamma-1}{\gamma^{t}-1}\right)$ and $\psi(t)=\log\left(\frac{\gamma-1}{\gamma^{1-t}-1}\right)$, The q-ROFAWHM operator will be changed into the q-ROF Frank weighted HM operator.

$$\mathrm{q-ROFAWHM}_{TF}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$= \left(\left(1-\log_\gamma\left(1+\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(\gamma\left(1+\left(\prod_{j=1}^{k}\left(\gamma\left(1+\frac{\left(\gamma^{1-\mu_{i_j}^q}-1\right)^{w_{i_j}}}{(\gamma-1)^{w_{i_j}-1}}\right)^{-1}-1\right)\right)^{\frac{1}{k}}\right)^{-1}-1\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{q}},$$

$$\left(\log_\gamma\left(1+\left(\prod_{1\le i_1<\ldots<i_k\le n}\left(\gamma\left(1+\left(\prod_{j=1}^{k}\left(\gamma\left(1+\frac{\left(\gamma^{\nu_{i_j}^q}-1\right)^{w_{i_j}}}{(\gamma-1)^{w_{i_j}-1}}\right)^{-1}-1\right)\right)^{\frac{1}{k}}\right)^{-1}-1\right)\right)^{\frac{1}{C_n^k}}\right)\right)^{\frac{1}{q}}\right) \tag{16}$$

Furthermore, if we select the parameter $k = n$ in the q-ROFAWHM operator, we will get the q-ROFAWG operator as follows [25]:

$$\mathrm{q-ROFAWHM}^{k=n}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\left(\phi^{-1}\left(\sum_{j=1}^{n} w_j\phi(\mu_j^q)\right)\right)^{\frac{1}{q}}, \left(\psi^{-1}\left(\sum_{j=1}^{n} w_j\psi(\nu_j^q)\right)\right)^{\frac{1}{q}}\right) \tag{17}$$

$$= \bigotimes_{j=1}^{n} \alpha_j^{w_j}$$

$$= \mathrm{q-ROFAWG}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

On the other hand, on selecting the parameter $k = 1$ in the q-ROFAWHM operator, we will get the following:

$$\mathrm{q-ROFAWHM}^{k=1}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \left(\left(\psi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} w_j\psi(\mu_j^q)\right)\right)^{\frac{1}{q}}, \left(\phi^{-1}\left(\frac{1}{n}\sum_{j=1}^{n} w_j\phi(\nu_j^q)\right)\right)^{\frac{1}{q}}\right) \tag{18}$$

$$= \frac{1}{n}\bigoplus_{j=1}^{n} w_j\alpha_j$$

It should be noted that the Equation 18 is 1/n times the q-ROF Archimedean weighted averaging (q-ROFAWA) operator that was developed in [25].

The q-ROFAWA operator defined in [25] is as follows:

$$\mathrm{q-ROFAWA}(\alpha_1,\alpha_2,\ldots,\alpha_n) = \bigoplus_{j=1}^{n} w_j\alpha_j$$

$$= \left(\left(\psi^{-1}\left(\sum_{j=1}^{n} w_j\psi(\mu_j^q)\right)\right)^{\frac{1}{q}}, \left(\phi^{-1}\left(\sum_{j=1}^{n} w_j\phi(\nu_j^q)\right)\right)^{\frac{1}{q}}\right) \tag{19}$$

Any specific pair of AGs of ATN and ATCN will provide that particular t-norm-based WA operator. To provide further insight into this framework, the resulting weighted AOs are discussed hereafter.

1. Algebraic: For $\phi(t^q) = -\log t^q$ and $\psi(t^q) = -\log(1-t^q)$. The q-ROFAWA operator is converted into the q-ROF WA operator [14].

$$\mathrm{q-ROFAWA}_{T^A}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \left(\left(1 - \prod_{j=1}^{n}\left(1-\mu_{\alpha_j}^q\right)^{w_j}\right)^{\frac{1}{q}}, \prod_{j=1}^{n}\nu_{\alpha_j}^{w_j}\right) \tag{20}$$

2. Aczél-Alsina: For $\phi(t^q) = (-\log t^q)^\gamma$ and $\psi(t^q) = (-\log(1-t^q))^\gamma$. The q-ROFAWA operator is reduced to the q-ROF Aczél-Alsina WA operator [26].

$$\begin{aligned}&\mathrm{q-ROFAWA}_{T^{AA}}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \left(\left(1 - e^{-\left(\sum_{j=1}^{n} w_j\left(-\log(1-\mu_{\alpha_j}^q)\right)^\gamma\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}, \left(e^{-\left(\sum_{j=1}^{n} w_j\left(-\log \nu_{\alpha_j}^q\right)^\gamma\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\right)\end{aligned} \tag{21}$$

3. Dombi: For $\phi(t^q) = \left(\frac{1-t^q}{t^q}\right)^\gamma$ and $\psi(t^q) = \left(\frac{t^q}{1-t^q}\right)^\gamma$. The q-ROFAWA operator is converted into the q-ROF Dombi WA operator [11].

$$\begin{aligned}&\mathrm{q-ROFAWA}_{T^D}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \left(\left(1 - \frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{u_{\alpha_j}^q}{1-\mu_{\alpha_j}^q}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}, \left(\frac{1}{1+\left(\sum_{j=1}^{n} w_j\left(\frac{1-\nu_{\alpha_j}^q}{\nu_{\alpha_j}^q}\right)^\gamma\right)^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\right)\end{aligned} \tag{22}$$

4. Frank: For $\phi(t^q) = \log\left(\frac{\gamma-1}{\gamma^{t^q}-1}\right)$ and $\psi(t^q) = \log\left(\frac{\gamma-1}{\gamma^{1-t^q}-1}\right)$. The q-ROFAWA operator is reduced to the q-ROF Frank WA operator [24].

$$\begin{aligned}&\mathrm{q-ROFAWA}_{T^F}(\alpha_1, \alpha_2, \ldots, \alpha_n)\\ &= \left(\left(1 - \log_\gamma\left(1 + \prod_{j=1}^{n}\left(\gamma^{1-\mu_{\alpha_j}^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}, \left(\log_\gamma\left(1 + \prod_{j=1}^{n}\left(\gamma^{\nu_{\alpha_j}^q}-1\right)^{w_j}\right)\right)^{\frac{1}{q}}\right)\end{aligned} \tag{23}$$

5. Hamacher: For $\phi(t^q) = \log\left(\frac{\gamma+(1-\gamma)t^q}{t^q}\right)$ and $\psi(t^q) = \log\left(\frac{\gamma+(1-\gamma)(1-t^q)}{1-t^q}\right)$. The q-ROFAWA operator is converted to the q-ROF Hamacher WA operator [6].

$$\mathrm{q}-\mathrm{ROFAWA}_{T^H}(\alpha_1,\alpha_2,\ldots,\alpha_n)$$

$$=\left(\left(\frac{\prod_{j=1}^{n}\left(1+(\gamma-1)\mu_{\alpha_j}^{q}\right)^{w_j}-\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{q})^{w_j}}{\prod_{j=1}^{n}\left(1+(\gamma-1)\mu_{\alpha_j}^{q}\right)^{w_j}+(\gamma-1)\prod_{j=1}^{n}(1-\mu_{\alpha_j}^{q})^{w_j}}\right)^{\frac{1}{q}},\frac{\gamma^{\frac{1}{q}}\prod_{j=1}^{n}(\nu_{\alpha_j})^{w_j}}{\left(\prod_{j=1}^{n}(1+(\gamma-1)(1-\nu_{\alpha_j}^{q}))^{w_j}+(\gamma-1)\prod_{j=1}^{n}(\nu_{\alpha_j}^{q})^{w_j}\right)^{\frac{1}{q}}}\right) \tag{24}$$

In the following section, we discuss a q-ROFAOPA method, as proposed in [21], for determining the weights of the decision attributes. In the subsequent sections, the q-ROFAOPA and q-ROFAWHM-operator-based MCGDM methodologies are proposed.

4. q-Rung Orthopair Fuzzy Archimedean Ordinal Priorities Approach

This section provides a concise explanation of the step-by-step procedure underlying the q-ROFAOPA method. This methodology is used to find the criteria weights, and it has three steps, which are as follows [21]:

Consider an MCGDM problem with n decision attributes and l DMs. Each expert expresses the relative importance of these n attributes through a linguistic assessment matrix $L^h=\left[\iota_{1j}^{h}\right]_{1\times n}$, where $j\in\mathbb{N}_n$, $h\in\mathbb{N}_l$. The term ι_{1j}^{h} denotes the relative significance assigned to the j^{th} attribute given by h^{th} expert.

Step 1. Choose a suitable linguistic q-ROF scale for constructing a q-ROF matrix $R=\left[\iota_{jh}\right]_{n\times l}$ of the n-attributes and l-experts.

Step 2. To fuse the assessments provided by the DMs for each attribute, apply the q-ROFAWA operator (see Equation 19). Next, for each aggregated value, compute the overall score (S_j) using Equation 4. The attributes are then ordered according to their corresponding score values.

Step 3. The weights of the attributes are required to satisfy the ordering constraint $w_j^{(t)}\geq w_j^{(t+1)}$, where $w_j^{(t)}$ represents the importance of the attribute occupying the t^{th} position in the ranking. This ordering condition leads to the following requirement:

$$\frac{\min\limits_{1\leq j\leq n} S_j}{S_j^{(t)}}\left(w_j^{(t)}-w_j^{(t+1)}\right)\geq 0 \tag{25}$$

On the basis of condition (25), the attribute weights can be computed by formulating the linear programming model shown in (26).

$$\begin{cases}\text{Max } Z\\ \text{s. t.}\\ \dfrac{\min\limits_{1\leq j\leq n} S_j}{S_j^{(t)}}\left(w_j^{(t)}-w_j^{(t+1)}\right)\geq Z;\ \dfrac{\min\limits_{1\leq j\leq n} S_j}{S_j^{(n)}}w_j^{(n)}\geq Z;\\ \sum\limits_{j=1}^{n} w_j=1;\ w_j\geq 0\,\forall\, j.\end{cases} \tag{26}$$

5. Multi-Criteria Group Decision-Making Approach

Suppose an MCGDM problem is defined over a q-ROFE, involving l DMs D_h, $h \in \mathbb{N}_l$, n attributes $C_j, j \in \mathbb{N}_n$, and m alternatives $A_i, i \in \mathbb{N}_m$. Let $w_j \in [0,1]$ denote the weight of criterion C_j, satisfying $\sum_{j=1}^{n} w_j = 1$ and let $\omega_h \in [0,1]$ represent the weight of an expert D_h with $\sum_{h=1}^{l} \omega_h = 1$. The evaluation of i^{th} alternatives corresponding to j^{th} criterion by h^{th} an expert is expressed as the ij^{th} element of the q-ROF decision matrix $M^h = \left[\alpha_{ij}^h\right]_{m\times n}$, where $\alpha_{ij}^h = \left(\mu_{ij}^h, \nu_{ij}^h\right)$ is a q-ROFN. Consequently, there exist l matrices M^1 to M^l. To identify the most suitable alternative, the proposed MCGDM model is applied as follows:

Step 1. Normalization

If an MCGDM problem involves both cost and benefit types of attributes. Then transformed the given decision matrices $M^h = \left[\alpha_{ij}^h\right]_{m\times n}$ into normalized decision matrices $\tilde{M}^h = \left[\tilde{\alpha}_{ij}^h\right]_{m\times n}$ by using the procedure (27):

$$\tilde{\alpha}_{ij}^h = \left(\tilde{\mu}_{ij}^h, \tilde{\nu}_{ij}^h\right) = \begin{cases} (\mu_{ij}^h, \nu_{ij}^h), & \text{for benefit type} \\ (\nu_{ij}^h, \mu_{ij}^h), & \text{for cost type} \end{cases} \tag{27}$$

Step 2. Decision matrices' aggregation

To fuse the normalized matrices $\tilde{M}^1$ to $\tilde{M}^l$ provided by experts, use the proposed q-ROFAWHM operator and the given experts' weights $(\omega_1$ to $\omega_l)$. This aggregation yields a collective decision matrix $\tilde{M} = [\tilde{\alpha}_{ij}]_{m\times n}$.

$$\tilde{\alpha}_{ij} = \text{q-ROFAWHM}(\tilde{\alpha}_{ij}^1, \tilde{\alpha}_{ij}^2, \ldots, \tilde{\alpha}_{ij}^l) \tag{28}$$

Step 3. Evaluation of criteria weights

Apply the q-ROFOPA method described in section 4 to find the importance degree for each decision criterion. The obtained weights w_1 to w_n will be used to compute the overall assessment values in the next step.

Step 4. Overall assessment values

Utilize the computed criteria weights (w_1 to w_n) to aggregate each row of the collective decision matrix ($\tilde{M}$) with the help of the q-ROFAWHM operator (12). Thus, for every alternative (A_1 to A_m), an overall assessment value ($\tilde{\alpha}_i$) is obtained as follows:

$$\tilde{\alpha}_i = \text{q-ROFAWHM}(\tilde{\alpha}_{i1}, \tilde{\alpha}_{i2}, \ldots, \tilde{\alpha}_{in}) \tag{29}$$

Step 5. Score and ranking

Using Equation 4, compute the score for all overall assessment values, i.e., $S(\tilde{\alpha}_i)$. Furthermore, use these computed values to assign a rank to each alternative.

5.1 *Practical Implementation of the Developed MCGDM Method*

Although mining operations are typically limited in duration, their impacts persist well beyond the active extraction phase. Effective closure and rehabilitation, therefore, become essential, as inadequate restoration can impose long-term burdens on society. Sustainable economic outcomes can be supported by diversifying the use of mining revenues, particularly through investments in physical, social, and human capital. In this way, the exploitation of non-renewable resources can remain consistent with sustainability principles, since depleted assets are transformed into forms of capital that continue to enhance societal well-being. Corporate social responsibility initiatives may involve allocating a portion of profits to community development, safeguarding employment, and improving workplace conditions. Assistance to regions affected by mine closure can further facilitate transitions to sectors such as agriculture, helping integrate mining activities within a broader sustainability framework. Following closure, a mining firm may consider several strategic options: rehabilitation (A_1), rehabilitation with business investment (A_2), and rehabilitation with social transition subsidy (A_3). Interviews with experts explored these choices, focusing on their implications for sustainability and incorporating a range of critical viewpoints. Criteria for socially responsible rehabilitation include four main attributes and twelve sub-attributes taken from [7]. The complete decision hierarchy of attributes, alternative assessments with respect to criteria, and significance of criteria evaluated by four DMs (D_1, D_2, D_3, D_4) are taken from [7] and shown in Table 3, Table 4, and Table 5, respectively.

Table 3: List of criteria involved in the problem [7]

Main Criteria	Sub-criteria	Type
($\mathcal{G}_1$) Economic aspect	(C_1) Income of the residents in the region	Benefit
	(C_2) Employment in the region	Benefit
	(C_3) Socially responsible activities	Cost
($\mathcal{G}_2$) Social aspect	(C_4) Migration to other cities	Cost
	(C_5) Social transition after the closure of a mine	Benefit
	(C_6) Social justice	Benefit
($\mathcal{G}_3$) Sustainability aspect	(C_7) The reputation of the mining company	Benefit
	(C_8) Social acceptance	Benefit
	(C_9) Providing sustainable land use	Benefit
($\mathcal{G}_4$) Environmental aspect	(C_{10}) Biodiversity	Benefit
	(C_{11}) GHG emissions	Cost
	(C_{12}) Contamination of soil	Cost

Table 4: Linguistic decision matrices provided by four DMs [7]

DM	Alternative	Criteria (C_j)											
D_h	A_i	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
D_1	A_1	L	ML	EH	EH	L	L	H	M	ML	MH	VL	L
	A_2	MH	H	MH	ML	M	M	EH	MH	H	MH	VL	L
	A_3	ML	L	ML	H	EH	MH	EH	H	EH	EH	VL	VL
D_2	A_1	VL	EL	EH	EH	L	EL	EH	M	M	VL	EL	EH
	A_2	H	EH	ML	EH	M	L	EH	M	EH	VL	EL	L
	A_3	M	ML	EL	EH	MH	L	EH	MH	EH	VL	EL	VL
D_3	A_1	VL	EL	EH	EH	EL	ML	H	MH	EH	EH	EL	EL
	A_2	EH	MH	VL	VL	EH	MH	EH	H	MH	M	ML	M
	A_3	H	MH	ML	VL	MH	EH	EH	EH	EH	EH	EL	VL
D_4	A_1	VL	L	M	H	ML	M	H	M	EH	EH	L	VL
	A_2	EH	H	L	VL	MH	H	EH	EH	M	MH	M	L
	A_3	EH	EH	EL	EL	EH	EH	EH	EH	H	EH	VL	EL

Table 5: Linguistic matrices of criteria significance provided by four DMs [7]

DM	Criteria ($\mathcal{G}$ & C)															
D_h	$\mathcal{G}_1$	$\mathcal{G}_2$	$\mathcal{G}_3$	$\mathcal{G}_4$	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
D_1	EH	MH	M	H	EL	H	MH	EL	M	VH	EH	H	H	VL	VH	H
D_2	EH	ML	M	MH	M	VH	H	ML	MH	EH	EH	MH	VH	L	VH	MH
D_3	VH	M	L	L	ML	MH	VL	VL	MH	ML	EH	L	M	VH	VH	H
D_4	EH	ML	M	MH	MH	H	ML	ML	VH	ML	EH	VH	ML	M	H	VH

Table 6 is used to convert the linguistic information of the selected problem into q-ROFNs, which is taken from [18].

Table 6: q-ROF linguistic scale [18]

q-ROFN (μ & ν)	Linguistic terms: Extremely low (EL)	Very low (VL)	Low (L)	Medium low (ML)	Medium (M)	Medium high (MH)	High (H)	Very high (VH)	Extremely high (EH)
Membership (μ)	0.15	0.25	0.35	0.45	0.55	0.65	0.75	0.85	0.95
Non-membership (ν)	0.95	0.85	0.75	0.65	0.55	0.45	0.35	0.25	0.15

Step 1. Normalization

Since all the criteria except C_3, C_4, C_{11}, and C_{12} are of benefit type, therefore employ the procedure (27) on the given decision matrices and obtain the normalized decision matrices $\tilde{M}^1$ to $\tilde{M}^4$, shown in Table 7.

Table 7: Normalized decision matrices

DM	Alternative	Criteria (C_j)											
D_h	A_i	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
	A_1	$(0.35, 0.75)$	$(0.45, 0.65)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.35, 0.75)$	$(0.35, 0.75)$	$(0.75, 0.35)$	$(0.55, 0.55)$	$(0.45, 0.65)$	$(0.65, 0.45)$	$(0.85, 0.25)$	$(0.75, 0.35)$
D_1	A_2	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.45, 0.65)$	$(0.65, 0.45)$	$(0.55, 0.55)$	$(0.55, 0.55)$	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.85, 0.25)$	$(0.75, 0.35)$
	A_3	$(0.45, 0.65)$	$(0.35, 0.75)$	$(0.65, 0.45)$	$(0.35, 0.75)$	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.95, 0.15)$	$(0.75, 0.35)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.85, 0.25)$	$(0.85, 0.25)$
	A_1	$(0.25, 0.85)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.35, 0.75)$	$(0.15, 0.95)$	$(0.95, 0.15)$	$(0.55, 0.55)$	$(0.55, 0.55)$	$(0.25, 0.85)$	$(0.95, 0.15)$	$(0.15, 0.95)$
D_2	A_2	$(0.75, 0.35)$	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.15, 0.95)$	$(0.55, 0.55)$	$(0.35, 0.75)$	$(0.95, 0.15)$	$(0.55, 0.55)$	$(0.95, 0.15)$	$(0.25, 0.85)$	$(0.95, 0.15)$	$(0.75, 0.35)$
	A_3	$(0.55, 0.55)$	$(0.45, 0.65)$	$(0.95, 0.15)$	$(0.15, 0.95)$	$(0.65, 0.45)$	$(0.35, 0.75)$	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.95, 0.15)$	$(0.25, 0.85)$	$(0.95, 0.15)$	$(0.85, 0.25)$
	A_1	$(0.25, 0.85)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.15, 0.95)$	$(0.45, 0.65)$	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$
D_3	A_2	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.85, 0.25)$	$(0.85, 0.25)$	$(0.95, 0.15)$	$(0.65, 0.45)$	$(0.95, 0.15)$	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.55, 0.55)$	$(0.65, 0.45)$	$(0.55, 0.55)$
	A_3	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.65, 0.45)$	$(0.85, 0.25)$	$(0.65, 0.45)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.85, 0.25)$
	A_1	$(0.25, 0.85)$	$(0.35, 0.75)$	$(0.55, 0.55)$	$(0.35, 0.75)$	$(0.45, 0.65)$	$(0.55, 0.55)$	$(0.75, 0.35)$	$(0.55, 0.55)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.75, 0.35)$	$(0.85, 0.25)$
D_4	A_2	$(0.95, 0.15)$	$(0.75, 0.35)$	$(0.75, 0.35)$	$(0.85, 0.25)$	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.55, 0.55)$	$(0.65, 0.45)$	$(0.55, 0.55)$	$(0.75, 0.35)$
	A_3	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.75, 0.35)$	$(0.95, 0.15)$	$(0.85, 0.25)$	$(0.95, 0.15)$

Step 2. Decision matrices' aggregation

To aggregate the above four decision matrices shown in Table 7, equal weights of DMs ($\omega_h = 0.25$ $\forall\ h$) is used in the q-ROFAWHM operator with $q = 2$, $k = 2$, and algebraic t-norm (T^A) and obtained a collective decision matrix $\tilde{M} = [\tilde{\alpha}_{ij}]_{3\times12}$, presented in Table 8.

Table 8: Aggregated decision matrix

Alternative	Criteria (C_j)					
A_i	C_1	C_2	C_3	C_4	C_5	C_6
A_1	$(0.1391, 0.9538)$	$(0.1357, 0.9619)$	$(0.1180, 0.9729)$	$(0.0980, 0.9788)$	$(0.1640, 0.9443)$	$(0.1906, 0.9315)$
A_2	$(0.5290, 0.7174)$	$(0.4691, 0.7519)$	$(0.3841, 0.8070)$	$(0.3622, 0.8417)$	$(0.3949, 0.8052)$	$(0.3123, 0.8534)$
A_3	$(0.3986, 0.8052)$	$(0.3459, 0.8423)$	$(0.5098, 0.7327)$	$(0.3445, 0.8639)$	$(0.5098, 0.7327)$	$(0.4627, 0.7754)$
Alternative	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
A_1	$(0.4884, 0.7374)$	$(0.3093, 0.8514)$	$(0.4599, 0.7748)$	$(0.4485, 0.7913)$	$(0.5712, 0.6845)$	$(0.4171, 0.8146)$
A_2	$(0.6642, 0.6223)$	$(0.4323, 0.7794)$	$(0.4323, 0.7794)$	$(0.2800, 0.8758)$	$(0.4564, 0.7646)$	$(0.3966, 0.7950)$
A_3	$(0.6642, 0.6223)$	$(0.5290, 0.7174)$	$(0.6073, 0.6633)$	$(0.5335, 0.7417)$	$(0.5941, 0.6666)$	$(0.5586, 0.6874)$

Likewise, the aggregated decision matrix can also be constructed to facilitate the final evaluation under various parametric t-norms (T_γ^{AA}, T_γ^{D}, T_γ^{F}, and T_γ^{H}) with parameter value $\gamma = 2$.

Step 3. Evaluation of criteria weights
Implement the q-ROFAOPA procedure as outlined in section 4.

Step 3.1. Apply the linguistic q-ROF scale provided in Table 6. The resulting q-ROF values are presented in Table 9.

Step 3.2. Aggregate the criteria importance values using the q-ROFAWA operator. Since the considered problem involves main and sub-criteria, to obtain the overall score values, first the score function shown in Equation 4 is used to compute the local scores of the criteria. Then, the overall or global score values (S_j) are obtained by multiplying the local score of each criterion by the local score of the corresponding main criterion. Lastly, rank the criteria by their global scores. The resultant aggregated values under T^A, global score and ranks are shown in Table 9.

Table 9: Significance and rank of criteria

Criteria	DMs (D_h)				Aggregated	Score		
($\mathcal{G}$ & C)	D_1	D_2	D_3	D_4	assessment	Local	Global	Rank
$\mathcal{G}_1$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.85, 0.25)$	$(0.95, 0.15)$	(0.9345, 0.1704)	0.9197		
C_1	$(0.95, 0.15)$	$(0.55, 0.55)$	$(0.45, 0.65)$	$(0.65, 0.45)$	(0.7611, 0.3941)	0.7055	0.6488	2
C_2	$(0.75, 0.35)$	$(0.85, 0.25)$	$(0.65, 0.45)$	$(0.75, 0.35)$	(0.7626, 0.3426)	0.7247	0.6665	1
C_3	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.85, 0.25)$	$(0.45, 0.65)$	(0.7167, 0.4000)	0.6690	0.6153	3
$\mathcal{G}_2$	$(0.65, 0.45)$	$(0.45, 0.65)$	$(0.55, 0.55)$	$(0.45, 0.65)$	(0.5372, 0.5687)	0.4734		
C_4	$(0.95, 0.15)$	$(0.45, 0.65)$	$(0.85, 0.25)$	$(0.45, 0.65)$	(0.7986, 0.3548)	0.7501	0.3551	10∼11
C_5	$(0.55, 0.55)$	$(0.65, 0.45)$	$(0.65, 0.45)$	$(0.85, 0.25)$	(0.7042, 0.4085)	0.6564	0.3107	12
C_6	$(0.85, 0.25)$	$(0.95, 0.15)$	$(0.45, 0.65)$	$(0.45, 0.65)$	(0.7986, 0.3548)	0.7501	0.3551	11∼10
$\mathcal{G}_3$	$(0.55, 0.55)$	$(0.55, 0.55)$	$(0.75, 0.35)$	$(0.55, 0.55)$	(0.6158, 0.4912)	0.5599		
C_7	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	$(0.95, 0.15)$	(0.9500, 0.1500)	0.9381	0.5252	5
C_8	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.85, 0.25)$	(0.7626, 0.3426)	0.7247	0.4058	8
C_9	$(0.75, 0.35)$	$(0.85, 0.25)$	$(0.55, 0.55)$	$(0.45, 0.65)$	(0.7002, 0.4206)	0.6486	0.3632	9
$\mathcal{G}_4$	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.65, 0.45)$	(0.7052, 0.3969)	0.6616		
C_{10}	$(0.85, 0.25)$	$(0.75, 0.35)$	$(0.85, 0.25)$	$(0.55, 0.55)$	(0.7800, 0.3312)	0.7425	0.4912	6
C_{11}	$(0.85, 0.25)$	$(0.85, 0.25)$	$(0.85, 0.25)$	$(0.75, 0.35)$	(0.8301, 0.2719)	0.8017	0.5304	4
C_{12}	$(0.75, 0.35)$	$(0.65, 0.45)$	$(0.75, 0.35)$	$(0.85, 0.25)$	(0.7626, 0.3426)	0.7247	0.4795	7

Step 3.3. To determine the criteria weights, formulate the linear programming model. To do so, incorporate the global scores and ranking information from Table 9 into the conditions specified in (26). The resulting model, given the information, is presented in (30).

$$\begin{cases} \text{Max } Z \\ \text{s. t.} \\ 0.4662\left(w_2^{(1)} - w_1^{(2)}\right) \geq Z;\ 0.5916\left(w_7^{(5)} - w_{10}^{(6)}\right) \geq Z; 0.8555\left(w_9^{(9)} - w_4^{(10)}\right) \geq Z; \\ 0.4789\left(w_1^{(2)} - w_3^{(3)}\right) \geq Z;\ 0.6325\left(w_{10}^{(6)} - w_{12}^{(7)}\right) \geq Z; 0.8750\left(w_4^{(10)} - w_6^{(11)}\right) \geq Z; \\ 0.5050\left(w_3^{(3)} - w_{11}^{(4)}\right) \geq Z;\ 0.6480\left(w_{12}^{(7)} - w_8^{(8)}\right) \geq Z; 0.8750\left(w_6^{(11)} - w_5^{(12)}\right) \geq Z; \\ 0.5858\left(w_{11}^{(4)} - w_7^{(5)}\right) \geq Z;\ 0.7656\left(w_8^{(8)} - w_9^{(9)}\right) \geq Z;\ 1.w_5^{(12)} \geq Z; \\ \sum_{j=1}^{12} w_j = 1;\ w_j \geq 0\, \forall\, j. \end{cases} \tag{30}$$

Solving the model (30) yields a maximum objective value of $Z = 0.0095$. The corresponding criteria weights are shown in Table 10 (see row first).

Table 10: Criteria weights under several t-norms

Triangular norm	Weights of criteria (w_j)											
	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}
Algebraic (T^A)	0.1560	0.1765	0.1361	0.0314	0.0095	0.0204	0.1009	0.0550	0.0425	0.0848	0.1172	0.0697
Aczél-Alsina (T_2^{AA})	0.1752	0.1542	0.1347	0.0441	0.0096	0.0326	0.1161	0.0564	0.0211	0.0850	0.1005	0.0705
Dombi (T_2^D)	0.1741	0.1521	0.1334	0.0456	0.0098	0.0337	0.1151	0.0578	0.0217	0.0854	0.0999	0.0713
Frank (T_2^F)	0.1561	0.1767	0.1363	0.0312	0.0095	0.0204	0.1010	0.0548	0.0423	0.0848	0.1174	0.0696
Hamacher (T_2^H)	0.1562	0.1769	0.1365	0.0310	0.0096	0.0203	0.1010	0.0546	0.0421	0.0847	0.1176	0.0695

In addition, Table 10 present the criteria weights obtained under several parametric t-norms (T_γ^{AA}, T_γ^D, T_γ^F, and T_γ^H) with $\gamma = 2$.

Step 4. Overall assessment values

Using the collective decision matrix $\tilde{M}_{3\times12}$ presented in Table 8 and the criteria weights under T^A shown in Table 10, the q-ROFAWHM operator is applied to compute the final decision values ($\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3$) for all alternatives (see Table 11).

Step 5. Score and ranking

Lastly, using the score function (Equation 4), the final score values ($S(\tilde{\alpha}_i)$) for all three alternatives and their ranks are presented in Table 11.

Table 11: Final results of the discussed MCGDM problem

T-norm	Alternative	Overall assessment	Score	Ranking order
Algebraic (T^A)	A_1	$(0.0833, 0.9910)$	0.0122	$A_3 > A_2 > A_1$
	A_2	$(0.1258, 0.9817)$	0.0255	
	A_3	$(0.1460, 0.9785)$	0.0314	
Aczél-Alsina (T_2^{AA})	A_1	$(0.2528, 0.9287)$	0.0989	$A_3 > A_2 > A_1$
	A_2	$(0.3300, 0.8786)$	0.1655	
	A_3	$(0.3877, 0.8549)$	0.2067	
Dombi (T_2^D)	A_1	$(0.3520, 0.7620)$	0.2644	$A_3 > A_2 > A_1$
	A_2	$(0.4032, 0.7054)$	0.3243	
	A_3	$(0.5519, 0.5623)$	0.4852	
Frank (T_2^F)	A_1	$(0.0795, 0.9921)$	0.0108	$A_3 > A_2 > A_1$
	A_2	$(0.1192, 0.9846)$	0.0220	
	A_3	$(0.1365, 0.9821)$	0.0266	
Hamacher (T_2^H)	A_1	$(0.0740, 0.9935)$	0.0090	$A_3 > A_2 > A_1$
	A_2	$(0.1096, 0.9878)$	0.0179	
	A_3	$(0.1239, 0.9859)$	0.0213	

Similarly, for various parametric t-norms (T_γ^{AA}, T_γ^D, T_γ^F, and T_γ^H) based collective decision matrix $\tilde{M}$ and corresponding criteria weights were also evaluated. The final assessment values, scores, and alternative rankings for these t-norms are presented in Table 11.

As shown in Table 11, the findings are both consistent and clear. Across all continuous ATNs considered, the developed methodology identifies A_3 as the most suitable alternative for a mining company to support community rehabilitation.

6. Conclusions

In summary, this work has employed the flexible structure of ATNs and ATCNs within the q-ROFE. Two AOs, q-ROFAHM and q-ROFAWHM, are introduced. These advancements culminate in a new MCGDM framework that offers a comprehensive solution for complex decision-making situations. The application of the proposed models to the selection of the best rehabilitation strategies for a closed mining site evaluated under various continuous ATNs indicates that the option incorporating a social transition subsidy (A_3) is the most viable choice for the mining company. Consequently, this study strengthens the practical utility of MCGDM in real-world settings.

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Conflicts of Interest

The authors declare no conflicts of interest.

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